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THE ANALYTIC PROPERTIES OF G_{2n} SPACES[1]

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ABSTRACT. A complex vector space X will be called an F_{2n} *space* if and only if there is a mapping $\langle\cdot,\cdots,\cdot\rangle$ from X^{2n} into the complex numbers such that: $\langle x,\cdots,x\rangle>0$ if $x\neq 0$; $x_k\to\langle x_1,\cdots,x_{2n}\rangle$ is linear for $k=1,\cdots,n$; $\langle x_1,\cdots,x_{2n}\rangle=\langle x_{2n},\cdots,x_1\rangle^-$ where $^-$ denotes complex conjugate; $\langle x_{\sigma(1)},\cdots,x_{\sigma(n)},y_{\tau(1)},\cdots,y_{\tau(n)}\rangle=\langle x_1,\cdots,x_n,y_1,\cdots,y_n\rangle$ for all permutations σ, τ of $\{1,\cdots,n\}$. In the case of a real vector space the mapping is assumed to be into the reals such that: $\langle x,\cdots,x\rangle>0$ if $x\neq 0$; $x_k\to\langle x_1,\cdots,x_{2n}\rangle$ is linear for $k=1,\cdots,2n$; $\langle x_{\sigma(1)},\cdots,x_{\sigma(2n)}\rangle=\langle x_1,\cdots,x_{2n}\rangle$ for all permutations σ of $\{1,\cdots,2n\}$. In either case, if $\|x\|=\langle x,\cdots,x\rangle^{1/2n}$ defines a norm, X is called a G_{2n} *space* (Trans. Amer. Math. Soc. **150** (1970), 507–518). It is shown that an F_{2n} space is a G_{2n} space if and only if $|\langle x,y,\cdots,y\rangle|^{2n}\leqq\langle x,\cdots,x\rangle\langle y,\cdots,y\rangle^{2n-1}$, and that G_{2n} spaces are examples of uniform semi-inner-product spaces studied by Giles (Trans. Amer. Math. Soc. **129** (1967), 436–446).

1. **Introduction.** Vector spaces equipped with multilinear functionals making them into F_{2n} spaces have been studied in **[2]**, **[4]**, **[6]**, **[8]**, and **[11]**. In the case that the functional satisfies the additional requirement

$$(1.1)\qquad |\langle x_1,\cdots,x_{2n}\rangle|^{2n}\leqq\prod_{i=1}^{n}\langle x_i,\cdots,x_i\rangle$$

it has been shown that the functional generates a norm in the same way a bilinear functional generates a norm on a Hilbert space, and that the associated normed linear space has many of the nice properties of an inner product space **[6**, p. 513, §3]. However, it is known that the functional may generate a norm without satisfying (1.1) **[6**, p. 513, Example 3.1].

The purpose of this paper is to show that if the functional generates a norm it must satisfy the weaker condition

$$(1.2)\qquad |\langle x,y,\cdots,y\rangle|^{2n}\leqq\langle x,\cdots,x\rangle\langle y,\cdots,y\rangle^{2n-1}$$

and that this condition is also sufficient to insure the same analytic properties as the stronger inequality (1.1).

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2. **The relationship between F_{2n} and G_{2n} spaces.** If X is an F_{2n} space and if we let

$$\|x\| = \langle x, \cdots, x\rangle^{1/2n}, \tag{2.1}$$

then $\|\cdot\|$ satisfies $\|x\| > 0$ if $x \neq 0$ and $\|tx\| = |t| \cdot \|x\|$ for all scalars t and all x in X. Thus to show that (2.1) defines a norm it is both necessary and sufficient to show that the triangle inequality is satisfied. Equivalently we must show that

$$g(t) = \|tx + (1 - t)y\|, \qquad t \text{ real}, \tag{2.2}$$

is a convex function of t for all x and y, and for this the inequality (1.2) is what is needed.

Observe that

$$h(t) = (g(t))^{2n} = \sum_{k=0}^{2n} \sum_{a=0}^{k} \sum_{r+s=a} \binom{n}{r}\binom{n}{s} A_{rs}(-1)^{k-a}\binom{2n-a}{2n-k} t^k \tag{2.3}$$

where $A_{rs} = \langle \overbrace{x, \cdots, x}^{r}, \overbrace{y, \cdots, y}^{n-r}, \overbrace{x, \cdots, x}^{s}, \overbrace{y, \cdots, y}^{n-s}\rangle$.

THEOREM (2.4). *An F_{2n} space is a G_{2n} space if and only if*

$$|\langle x, y, \cdots, y\rangle|^{2n} \leqq \langle x, \cdots, x\rangle\langle y, \cdots, y\rangle^{2n-1}$$

for all x and y.

PROOF. Assume $\|\cdot\|$ is a norm. Then $g(t) = \|tx + (1-t)y\|$, and hence $h(t) = (g(t))^{2n}$ is a convex function of t. Because of the homogeneity of the functional we need only establish the inequality when $\|x\| = \|y\| = 1$. In this case $h(0) = h(1) = 1$, and since h is a convex polynomial it follows that $h'(0) \leqq 0$. From (2.3) we can see that

$$\begin{aligned} h'(0) &= nA_{10} + nA_{01} - 2nA_{00} \\ &= 2n \operatorname{Re} A_{10} - 2nA_{00}. \end{aligned}$$

Therefore $\operatorname{Re}\langle x, y, \cdots, y\rangle \leqq \langle y, \cdots, y\rangle = 1$. But by using the homogeneity again we can easily obtain $|\langle x, y, \cdots, y\rangle| \leqq 1$ which is equivalent to the desired inequality when $\|x\| = \|y\| = 1$.

Conversely, if we assume the inequality holds we can use it exactly as Hölder's inequality to see that

$$\begin{aligned} \|x + y\|^{2n} &= \langle x + y, \cdots, x + y\rangle \\ &= \langle x, x + y, \cdots, x + y\rangle + \langle y, x + y, \cdots, x + y\rangle \\ &\leqq \langle x, \cdots, x\rangle^{1/2n}\langle x + y, \cdots, x + y\rangle^{(2n-1)/2n} \\ &\quad + \langle y, \cdots, y\rangle^{1/2n}\langle x + y, \cdots, x + y\rangle^{(2n-1)/2n} \\ &= (\|x\| + \|y\|)\, \|x + y\|^{2n-1}. \end{aligned}$$

But this is equivalent to the triangle inequality.

We should note that if we define

$$(2.5)\qquad \begin{aligned} &[x, y] = \langle x, y, \cdots, y\rangle / \langle y, \cdots, y\rangle^{(2n-2)/2n} && \text{if } y \neq 0,\\ &[x, y] = 0 && \text{if } y = 0,\end{aligned}$$

then the inequality (1.2) is equivalent to (2.5) defining a semi-inner-product. (See [7] for a discussion of semi-inner-products.) Thus we have an immediate corollary to the theorem.

COROLLARY (2.6). *An F_{2n} space is a G_{2n} space if and only if it is a semi-inner-product space with the semi-inner-product given by* (2.5). *The norm generated by the semi-inner-product is the same as the norm on the G_{2n} space.*

3. **The analytic properties of G_{2n} spaces.** We now turn to a consideration of the analytic properties of G_{2n} spaces. We shall show that they are uniformly convex, uniformly continuous semi-inner-product spaces. This kind of space has been studied by Giles [3], and we can conclude from his work that G_{2n} spaces are smooth, and in fact, if they are complete, every continuous linear functional can be represented uniquely in the form $\langle \cdot, y, \cdots, y\rangle$. In addition, because of the uniform convexity, it follows from a theorem of Millman and Pettis [10] that complete G_{2n} spaces are reflexive. It is interesting that this fact does not seem to follow from the representation theorem for continuous linear functionals in terms of the multilinear functional as it does in the case of a Hilbert space. Finally, orthogonality as studied by James (y is orthogonal to x if and only if $\|x\| \leqq \|x+\lambda y\|$ for all scalars λ [5]) is equivalent to orthogonality in terms of the multilinear functional (y is orthogonal to x if and only if $\langle x, y, \cdots, y\rangle = 0$).

We begin by establishing that G_{2n} spaces are uniformly continuous semi-inner-product spaces. To do this we must show that

$$(3.1)\qquad \lim_{\lambda\to 0} \operatorname{Re}[y, x+\lambda y] = \operatorname{Re}[y, x], \qquad \lambda \text{ real},$$

where the limit is uniform for $\|x\| = \|y\| = 1$ [3, p. 437]. Here $[\cdot, \cdot]$ is the semi-inner-product defined by (2.5). The result is clear if either x or y is the zero vector and so it is sufficient to consider the case when neither of them is zero. Thus we need to show that

$$(3.2)\qquad \begin{aligned} &\lim_{\lambda\to 0} \langle y, x+\lambda y, \cdots, x+\lambda y\rangle / \langle x+\lambda y, \cdots, x+\lambda y\rangle^{(2n-2)/2n} \\ &\qquad\qquad = \langle y, x, \cdots, x\rangle / \langle x, \cdots, x\rangle^{(2n-2)/2n}, \end{aligned}$$

but this is a consequence of the continuity of the multilinear functional [6, p. 511, Corollary (2.5)].

To see that the limit is uniform when $\|x\|=\|y\|=1$ we first need the following lemma which is an immediate corollary to the polarization identity for multilinear functionals [**6**, p. 510, Theorem (2.1)].

LEMMA (3.3). *If X is a G_{2n} space then for $\|x_1\|=\cdots=\|x_{2n}\|=1$, $|\langle x_1, \cdots, x_{2n}\rangle|\leqq(n!)^{-2}(2n)^{2n}$.*

If we use the notation of §2 we have

$$\operatorname{Re}\langle y, x+\lambda y, \cdots, x+\lambda y\rangle = \operatorname{Re}\langle y, x, \cdots, x\rangle + \lambda H(x, y, \lambda)$$

where

$$H(x, y, \lambda) = \sum_{r+s<2n-1} \binom{n-1}{r}\binom{n}{s} \lambda^{2n-r-s-2} \operatorname{Re} A_{rs}.$$

But it follows from Lemma (3.3) that if $\|x\|=\|y\|=1$ and $|\lambda|\leqq 1$ then $H(x, y, \lambda)$ is uniformly bounded. Hence $\lim_{\lambda\to 0}\langle y, x+\lambda y, \cdots, x+\lambda y\rangle = \langle y, x, \cdots, x\rangle$ is a uniform limit for $\|x\|=\|y\|=1$.

Finally $\langle x+\lambda y, \cdots, x+\lambda y\rangle^{(2n-2)/2n}=\|x+\lambda y\|^{2n-2}$ is a uniformly continuous function of λ for $\|x\|=\|y\|=1$ and is uniformly bounded away from 0 for $|\lambda|\leqq\frac{1}{2}$. It follows that the limit in (3.2), and thus in (3.1), is uniform for $\|x\|=\|y\|=1$.

THEOREM (3.4). *A G_{2n} space is a uniformly continuous semi-inner-product space.*

We next establish that G_{2n} spaces are uniformly convex.

THEOREM (3.5). *A G_{2n} space is uniformly convex.*

PROOF. We must show that if we have two sequences of unit vectors $\{x_j\}$, $\{y_j\}$ such that $\|(x_j+y_j)/2\|\to 1$ then $\|x_j-y_j\|\to 0$. As in §2 we shall let

$$h_j(t) = \|tx_j + (1-t)y_j\|^{2n}.$$

Then $\{h_j\}$ is a sequence of convex polynomials of degree $2n$ such that $h_j(0)=h_j(1)=1$, $h_j(\frac{1}{2})\to 1$, and $h_j(t)\leqq 1$ for all t in $[0, 1]$. As a result $h_j\to 1$ uniformly on $[0, 1]$, and thus the coefficients of t^k in h_j converge to 0 if $1\leqq k\leqq 2n$ and 1 if $k=0$. But from (2.3) we can see that the coefficient of t^k in h_j is

$$\sum_{a=0}^{k}\sum_{r+s=a}\binom{n}{r}\binom{n}{s}A_{rs}(j)(-1)^{k-a}\binom{2n-a}{2n-k} \tag{3.6}$$

where

$$A_{rs}(j) = \langle \overbrace{x_j, \cdots, x_j}^{r}, \overbrace{y_j, \cdots, y_j}^{n-r}, \overbrace{x_j, \cdots, x_j}^{s}, \overbrace{y_j, \cdots, y_j}^{n-s}\rangle.$$

To complete the proof of the theorem we need the following lemma.

LEMMA (3.7). *With x_j and y_j as in the proof of the theorem,*

$$\lim_{j\to\infty}\sum_{r+s=a}\binom{n}{r}\binom{n}{s}A_{rs}(j)=\binom{2n}{a}\quad \textit{for } a=0,\cdots,2n.$$

PROOF. The proof is by induction on a. For $a=0$ we have

$$\lim_{j\to\infty}\sum_{r+s=0}\binom{n}{r}\binom{n}{s}A_{rs}(j)=\lim_{j\to+\infty}A_{00}(j)=\lim_{j\to\infty}\langle y_j,\cdots,y_j\rangle=1=\binom{2n}{0}.$$

Now assume that the result is true for $a\leqq k$. Since the coefficients of t^{k+1} in h_j converge to 0 we have from (3.6) that

$$\lim_{j\to\infty}\sum_{a=0}^{k+1}\sum_{r+s=a}\binom{n}{r}\binom{n}{s}A_{rs}(j)(-1)^{k+1-a}\binom{2n-a}{2n-(k+1)}=0.$$

Then by the inductive assumption it follows that $\lim_{j\to\infty}\sum_{r+s=k+1}\binom{n}{r}\binom{n}{s}A_{rs}(j)$ exists and is equal to

$$\begin{aligned}-\lim_{j\to\infty}\sum_{a=0}^{k}\sum_{r+s=a}\binom{n}{r}\binom{n}{s}A_{rs}(j)(-1)^{k+1-a}\binom{2n-a}{2n-(k+1)}\\ =-\sum_{a=0}^{k}\binom{2n}{a}(-1)^{k+1-a}\binom{2n-a}{2n-(k+1)}.\end{aligned}$$

But a simple computation shows this last sum is $\binom{2n}{k+1}$ which completes the proof of the lemma.

To complete the proof of the theorem we need only observe that

$$\begin{aligned}\lim_{j\to\infty}\|x_j-y_j\|^{2n}&=\lim_{j\to\infty}\langle x_j-y_j,\cdots,x_j-y_j\rangle\\ &=\lim_{j\to\infty}\sum_{r=0}^{n}\sum_{s=0}^{n}\binom{n}{r}\binom{n}{s}(-1)^{2n-(r+s)}A_{rs}(j)\\ &=\lim_{j\to\infty}\sum_{a=0}^{2n}(-1)^{2n-a}\sum_{r+s=a}\binom{n}{r}\binom{n}{s}A_{rs}(j)\\ &=\sum_{a=0}^{2n}(-1)^{2n-a}\binom{2n}{a}=0.\end{aligned}$$

We conclude our development by strengthening Theorem (2.4).

COROLLARY (3.8). *If X is a G_{2n} space then equality holds in*

$$|\langle x,y,\cdots,y\rangle|^{2n}\leqq\langle x,\cdots,x\rangle\langle y,\cdots,y\rangle^{2n-1}$$

if and only if x and y are linearly dependent.

PROOF. If $y \neq 0$ the inequality is equivalent to the semi-inner-product inequality

$$|[x, y]| \leqq \|x\| \cdot \|y\|.$$

Since G_{2n} spaces are uniformly convex, and uniform convexity implies strict convexity, our corollary follows from [**1**, p. 381].

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