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ON CONVEX SUBSETS OF A POLYTOPE

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ABSTRACT. A. J. Hoffman conjectured the following: Given a d-polytope P and a collection, $C_1, \cdots, C_k$, of closed convex subsets of P with the property that each t-flat, $0 \leqq t \leqq d-1$, which meets P also meets some C_i, then there exist polytopes $D_j \subset C_j$ such that every t-flat which meets P also meets some D_j. In this note it is shown that the above is true for $k=2$.

In [2], Hoffman stated the following *conjecture* $C(d, t, k)$: If P is a d-polytope, $d \geqq 1$, and $t \geqq 0$ and $k \geqq 1$ are integers, $C_1, \cdots, C_k$ are closed convex sets in P such that every t-flat which meets P meets $\bigcup_{i=1}^{k} C_i$, then there are polytopes $D_1, \cdots, D_k$ with $D_i \subset C_i$, $1 \leqq i \leqq k$, such that every t-flat which meets P meets $\bigcup_{i=1}^{k} D_i$ also. He established $C(d, 0, k)$ in this same paper. Zaks ([3], [4], [5]) has shown that $C(d, d-2, k)$, $d \geqq 3$, $k \geqq 4$, is false, that each of $C(d, d-1, k)$ for all d and k, $C(d, t, 1)$ and $C(3, 1, 3)$ is true.

The purpose of this note is to prove $C(d, t, 2)$ for which two lemmas are established and the resulting theorem follows. (For notation, see Grünbaum [1].)

LEMMA 1. *Let P be a d-polytope and $C_1, \cdots, C_k$ be closed convex subsets of P such that every t-flat that meets P meets $\bigcup_{i=1}^{k} C_i$, $0 \leqq t \leqq d-2$; then* $\operatorname{skel}_m P \subset \bigcup_{i=1}^{k} C_i$, *where $m = d-t-1$.*

PROOF. Let F be an m-face of P, and let H be a supporting hyperplane of P such that $H \cap P = F$; let x be an arbitrary point of F. The affine flat in H, orthogonal to the affine hull of F and passing through x, is of dimension $d-1-m = d-1-(d-t-1) = t$; it meets P in exactly $\{x\}$; therefore by the assumption on $C_1, \cdots, C_k$, we have $x \in \bigcup_{i=1}^{k} C_i$ as promised.

LEMMA 2. *Suppose P is a d-polytope and C_1 and C_2 are closed convex subsets of P such that* $\operatorname{skel}_1 P \subset C_1 \cup C_2$; *then $P = C_1 \cup C_2$.*

PROOF. If $P \neq C_1 \cup C_2$, then there is a face F of lowest dimension $m \geqq 2$ such that $F \not\subset C_1 \cup C_2$. By minimality of m, $\operatorname{bd}(F) \subset C_1 \cup C_2$.

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Let $p \in \text{rel int}(F)\backslash(C_1 \cup C_2)$ and consider the mapping $x \to x'$ defined as follows. For $x \in \text{bd}(F)$, let x' be the (unique) point of $\text{bd}(F)$ such that the line $L(p, x)$ meets F in the segment xx'. By the choice of p, it follows that x and x' are in distinct members of $\{C_1, C_2\}$. Now

$$\text{bd}(F) = (\text{bd}(F) \cap C_1) \cup (\text{bd}(F) \cap C_2)$$

and $(\text{bd}(F) \cap C_1) \cap (\text{bd}(F) \cap C_2) = \varnothing$ with $\text{bd}(F) \cap C_1$ and $\text{bd}(F) \cap C_2$ nonempty. Since C_1 and C_2 are closed, this contradicts the connectedness of $\text{bd}(F)$.

THEOREM. *Let P be a d-polytope and let C_1 and C_2 be closed convex subsets of P such that every t-flat, $0 \leqq t \leqq d-1$, which meets P also meets $C_1 \cup C_2$. Then there exists polytopes $D_1 \subset C_1$, $D_2 \subset C_2$ such that every t-flat which meets P meets $D_1 \cup D_2$.*

PROOF. From the lemmas, $P = C_1 \cup C_2$; hence Hoffman's theorem in [2] is sufficient to complete the proof.

The portions of $C(d, t, k)$ which remain open are for $0 < t \leqq d-3$ with $d \geqq 4$ and $k \geqq 3$.

REFERENCES

1. B. Grünbaum, *Convex polytopes*, Pure and Appl. Math., vol. 16, Interscience, New York, 1967. MR **37** #2085.

2. A. J. Hoffman, *On covering of polyhedra by polyhedra*, Proc. Amer. Math. Soc. **23** (1969), 123–126. MR **40** #835.

3. J. Zaks, *On a conjecture of A. J. Hoffman*, Proc. Amer. Math. Soc. **27** (1971), 122–125.

4. ———, *On a conjecture of A. J. Hoffman*. II, Proc. Amer. Math. Soc. (to appear).

5. ———, *On Hoffman's conjecture* (manuscript).

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