ON CONVEX SUBSETS OF A POLYTOPE

W. R. HARE, JR. AND C. R. SMITH

Abstract. A. J. Hoffman conjectured the following: Given a d-polytope \( P \) and a collection, \( C_1, \ldots, C_k \), of closed convex subsets of \( P \) with the property that each t-flat, \( 0 \leq t \leq d-1 \), which meets \( P \) also meets some \( C_i \), then there exist polytopes \( D_1 \subset C_i \) such that every t-flat which meets \( P \) also meets some \( D_i \). In this note it is shown that the above is true for \( k=2 \).

In [2], Hoffman stated the following conjecture \( C(d, t, k) \): If \( P \) is a d-polytope, \( d \geq 1 \), and \( t \geq 0 \) and \( k \geq 1 \) are integers, \( C_1, \ldots, C_k \) are closed convex sets in \( P \) such that every t-flat which meets \( P \) meets \( \bigcup_{i=1}^{k} C_i \), then there are polytopes \( D_1, \ldots, D_k \) with \( D_i \subset C_i \), \( 1 \leq i \leq k \), such that every t-flat which meets \( P \) meets \( \bigcup_{i=1}^{k} D_i \) also. He established \( C(d, 0, k) \) in this same paper. Zaks ([3], [4], [5]) has shown that \( C(d, d-2, k) \), \( d \geq 3 \), \( k \geq 4 \), is false, that each of \( C(d, d-1, k) \) for all \( d \) and \( k \), \( C(d, t, 1) \) and \( C(3, 1, 3) \) is true.

The purpose of this note is to prove \( C(d, t, 2) \) for which two lemmas are established and the resulting theorem follows. (For notation, see Grünbaum [1].)

**Lemma 1.** Let \( P \) be a d-polytope and \( C_1, \ldots, C_k \) be closed convex subsets of \( P \) such that every t-flat that meets \( P \) meets \( \bigcup_{i=1}^{k} C_i \), \( 0 \leq t \leq d-2 \); then \( \text{skel}_m P \subset \bigcup_{i=1}^{k} C_i \), where \( m = d-t-1 \).

**Proof.** Let \( F \) be an \( m \)-face of \( P \) and let \( H \) be a supporting hyperplane of \( P \) such that \( H \cap P = F \); let \( x \) be an arbitrary point of \( F \). The affine flat in \( H \), orthogonal to the affine hull of \( F \) and passing through \( x \), is of dimension \( d-1-m = d-1-(d-t-1) = t \); it meets \( P \) in exactly \( \{x\} \); therefore by the assumption on \( C_1, \ldots, C_k \), we have \( x \in \bigcup_{i=1}^{k} C_i \) as promised.

**Lemma 2.** Suppose \( P \) is a d-polytope and \( C_1 \) and \( C_2 \) are closed convex subsets of \( P \) such that \( \text{skel}_m P \subset C_1 \cup C_2 \); then \( P = C_1 \cup C_2 \).

**Proof.** If \( P \not\subset C_1 \cup C_2 \), then there is a face \( F \) of lowest dimension \( m \geq 2 \) such that \( F \subset C_1 \cup C_2 \). By minimality of \( m \), \( \text{bd}(F) \subset C_1 \cup C_2 \).

Received by the editors May 19, 1971 and, in revised form, November 18, 1971.

AMS 1969 subject classifications. Primary 5210, 5230; Secondary 5090.

Key words and phrases. Convex polytope, closed convex set, affine flat, convex hull.

\(^1\) Latter author supported by NSF Fellowship No. 60175.
Let $p \in \text{rel int}(F) \setminus (C_1 \cup C_2)$ and consider the mapping $x \rightarrow x'$ defined as follows. For $x \in \text{bd}(F)$, let $x'$ be the (unique) point of $\text{bd}(F)$ such that the line $L(p, x)$ meets $F$ in the segment $xx'$. By the choice of $p$, it follows that $x$ and $x'$ are in distinct members of $\{C_1, C_2\}$. Now
\[ \text{bd}(F) = (\text{bd}(F) \cap C_1) \cup (\text{bd}(F) \cap C_2) \]
and $(\text{bd}(F) \cap C_1) \cap (\text{bd}(F) \cap C_2) = \emptyset$ with $\text{bd}(F) \cap C_1$ and $\text{bd}(F) \cap C_2$ nonempty. Since $C_1$ and $C_2$ are closed, this contradicts the connectedness of $\text{bd}(F)$.

**Theorem.** Let $P$ be a $d$-polytope and let $C_1$ and $C_2$ be closed convex subsets of $P$ such that every $t$-flat, $0 \leq t \leq d-1$, which meets $P$ also meets $C_1 \cup C_2$. Then there exists polytopes $D_1 \subset C_1$, $D_2 \subset C_2$ such that every $t$-flat which meets $P$ meets $D_1 \cup D_2$.

**Proof.** From the lemmas, $P = C_1 \cup C_2$; hence Hoffman's theorem in [2] is sufficient to complete the proof.

The portions of $C(d, t, k)$ which remain open are for $0 < t \leq d-3$ with $d \geq 4$ and $k \geq 3$.

**References**

5. ———, *On Hoffman's conjecture* (manuscript).

Department of Mathematics, Clemson University, Clemson, South Carolina 29631

Department of Mathematics, Northeast Louisiana University, Monroe, Louisiana 71201

Department of Mathematics, University of Washington, Seattle, Washington 98105