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SECOND ORDER ALMOST LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS—OSCILLATION

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ABSTRACT. It is shown that all solutions of certain second order nonlinear functional differential equations are oscillatory if all solutions of an associated minorizing linear equation are oscillatory.

1. **Introduction.** This note deals with the oscillation of all solutions of equations

$$x''(t) + F(t, x(t), x(t - \tau(t))) = 0 \tag{1}$$

where

(i) $F(t, u, v) \in C([0, \infty) \times R \times R)$, $uv > 0$ implies $F(t, u, v)$ is nondecreasing in u and v, $c \neq 0$ implies $\int^{\infty} csF(s, c, c)\, ds = \infty$;

(ii) $\tau(t) \in C[0, \infty)$, lim sup $\tau(t) = \tau_0 < \infty$ as $t \to \infty$; and

(iii) there exists a nonnegative function $a(t) \in C[0, \infty)$ such that for some $X_0 \geqq 0$ and some $\varepsilon > 0$, $|x| \geqq X_0$ implies $(1+\varepsilon)a(t)x^2 \leqq xF(t, x, x)$, while all solutions of

$$y'' + a(t)y = 0 \tag{2}$$

are oscillatory. In particular, the result holds for the linear equation $x''(t) + a(t)x(t - \tau(t)) = 0$.

In the ensuing the term *solution* refers only to those solutions of (1) which exist on some positive half-line. A solution of (1) or (2) is *oscillatory* if it has a zero in each positive half-line. The result to be proven is

THEOREM. *If* (1) *satisfies* (i), (ii), (iii), *then every solution of* (1) *is oscillatory. If, for all large t-values,* $\tau(t) \leqq 0$, *then the result holds if* $\varepsilon = 0$.

2. **Preliminaries.** Part A of the proof is related to a result of Ladas **[2]** and holds even if $\tau_0 = \infty$ provided $t - \tau(t) \to \infty$ as $t \to \infty$. Note that $\int^{\infty} sa(s)\, ds = \infty$ is necessary for the oscillation of all solutions of (2) **[7]**.

Part B makes use of a special case of Grimmer's and Waltman's **[1]** generalization of the Sturm Comparison Theorem.

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COMPARISON THEOREM. *Let* $y(t)$ *satisfy* (2) *and let* $x(t)$ *satisfy*

$$x'' + a(t)x \leqq 0, \qquad x > 0, \tag{3}$$

on (t_0, t_1) *with* $y(t_0) \geqq x(t_0) \geqq 0$, *and* $y'(t_0) \geqq x'(t_0) \geqq 0$, *but not both* $x(t_0) = x'(t_0) = 0$. *Then,* $y(t) \geqq x(t)$ *on* $[t_0, t_1]$.

Nonoscillation criteria for the linear case of (1) are discussed by Shere in [3].

3. **The proof.** It is clear that if $x(t)$ is a nonoscillatory solution of (1), then, for any $\tau_1 > \tau_0$, $x(t)$ may be assumed positive on some interval $[T-\tau_1, \infty)$. Therefore, $x''(t) \leqq 0$ $(\not\equiv 0)$ on $[T, \infty)$, wherefrom follows $x'(t) > 0$ on the same interval.

A. Certainly there is a $c > 0$ such that $x(t) > c$ on $[T, \infty)$ and $x(t)$ satisfies $x''(t) + F(t, c, c) \leqq 0$.

After a multiplication by t an integration by parts gives

$$tx'(t) - x(t) + C + \int_T^t sF(s, c, c)\, ds \leqq 0, \qquad t > T, \tag{4}$$

where C is a constant. If $x(t)$ is bounded above, (i) is contradicted if t is large enough.

B. Hence, assume $x(t) \to \infty$ as $t \to \infty$; and, it may be assumed that $x(t) > X_0$ on $[T, \infty)$. Let the positive ε of (iii) be given.

Integration of $x''(t) \leqq 0$ leads to $x(t) - x(t-\tau(t)) \leqq x'(T)\tau_1$. Thus, there is a T_1 so large that

$$(1 + \varepsilon)^{-1} x(t) \leqq x(t - \tau(t)), \qquad t \geqq T_1; \tag{5}$$

and $x(t)$ must satisfy (3) on $[T_1, \infty)$. Evidently, $\varepsilon = 0$ is sufficient if $\tau(t) \leqq 0$ on some $[T_1, \infty)$.

Therefore, the Comparison Theorem implies there is an oscillatory solution of (2) which majorizes $x(t)$ on some positive half-line. There is an obvious contradiction, and the theorem is proved.

REMARKS. (1) This result improves [**4**], and even in the case $\tau(t) \equiv 0$, appears to be new.

(2) P. Waltman has shown [**6**] how a $\tau(t)$ which is not bounded above influences oscillation.

(3) The theorem could have been given for more general equations (1) having more than one functional argument which may depend explicitly on x, x', in certain ways. Also, F need not necessarily be nondecreasing in its x-arguments ([**4**], [**5**]).

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