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A NOTE ON SOME CHARACTERIZATIONS OF SIDON SETS

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ABSTRACT. A simple proof of the following theorem is presented: $E\subset \boldsymbol{Z}$ is a Sidon set if and only if $A_E(T)$ equals any one of the following three spaces: $U_E(T)$, $\bigcap_{p>1} A_E^p(T)$, and $C_E(T, \omega)$.

In this note we present an elementary proof of some characterizations of Sidon sets, announced by K. Ross (joint work with R. E. Edwards and E. Hewitt) at the International Conference on Harmonic Analysis in Maryland, November 1971 (cf. [**1**]).

We refer to [**2**] for standard definitions and facts. For $E\subset \boldsymbol{Z}$, let $U_E(T)=\{f\in C_E(T):\langle S_n(f)\rangle$ converges uniformly to $f\}$, and let $A_E^p(T)=\{f\in C_E(T):\hat{f}\in l^p\}$; for a fixed element $\omega\in c_0(E)$, let $A_E(T,\omega)=\{f\in C_E(T):\omega\hat{f}\in l^1\}$. $E\subset \boldsymbol{Z}$ is a Sidon set if $A_E(T)=C_E(T)$. We remark that E is a Sidon set if and only if $\inf\{\|f\|_\infty/\|f\|_A : f\in A_E, f\neq 0\}>0$.

THEOREM. *The following statements are equivalent*:

(i) *E is a Sidon set*,

(ii) $U_E(T)=A_E(T)$,

(iii) $\bigcap_{p>1} A_E^p(T)=A_E(T)$,

(iv) $A_E(T,\omega)=A_E(T)$.

PROOF. (ii), (iii)$\Rightarrow$(i): Suppose $E\subset \boldsymbol{Z}$ is not a Sidon set. Without loss of generality, we assume $E\subset \boldsymbol{Z}^+$. By our preceding remark, we select $\langle f_j\rangle$, a sequence of trigonometric polynomials in $C_E(T)$, subject to the following conditions:

(i) $\|f_j\|_\infty<1/2^j$,

(ii) $\|f_j\|_A=1/j$,

(iii) $N_j\equiv\min\{n:n\in \text{support } (\hat{f}_j)\}>\max\{n:n\in\text{support } (\hat{f}_i)\}$, whenever $j>i$.

Set $f=\sum f_j$. Clearly $f\in C_E$, and, by (ii), $f\notin A_E$.

By (i), $\langle S_{N_j}(f)\rangle$ converges uniformly to f. It follows from (ii) that

$$\begin{aligned}\|S_N(f)-f\|_\infty &= \|S_{N_j}(f)+(S_N(f)-S_{N_j}(f))-f\|_\infty\\ &\leq \|S_{N_j}(f)-f\|_\infty+\|S_N(f)-S_{N_j}(f)\|_\infty\\ &\leq \|S_{N_j}(f)-f\|_\infty+1/j,\end{aligned}$$

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where j is selected so that $N_j < N < N_{j+1}$. Therefore, $f \in U_E$. Since $\|f_j\|_A \geqq \|\hat{f}_j\|_p$, and $\sum \|f_j\|_A^p < \infty$ for all $p > 1$, we have $f \in \bigcap_{p>1} A_E^p$.

(iv)$\Rightarrow$(i). Let $\langle i_j \rangle$ be so that $\omega_k < 1/j$ for all $k \geqq i_j$. Select f_j as above, with the additional requirement that $N_j \geqq i_j$. It easily follows that $\omega \hat{f} \in l^1$. The theorem is proved.

REMARK. The proof of (iii), (iv)$\Leftrightarrow$(i) can easily be adapted to general compact abelian groups.

REFERENCES

1. R. E. Edwards, E. Hewitt and K. A. Ross, *Lacunarity for compact groups*. II, Pacific J. Math. (to appear).

2. J.-P. Kahane, *Séries de Fourier absolument convergentes*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 50, Springer-Verlag, Berlin and New York, 1970. MR **43** #801.

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