PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, No. 1, September 1972

ON A GENERAL RATIO ERGODIC THEOREM WITH WEIGHTED AVERAGES

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ABSTRACT. A general ratio ergodic theorem with weighted averages is shown by utilizing a method of R. V. Chacon. The theorem contains Chacon's general ergodic theorem as a special case.

Let $(X, \mathcal{M}, m)$ be a σ-finite measure space and let T be a linear contraction on $L^1(m)$. Let $\{w_n; n \geqq 1\}$ be a sequence of nonnegative numbers whose sum is one, and let $\{u_n; n \geqq 0\}$ be the sequence defined by

$$u_n = w_1 u_{n-1} + \cdots + w_n u_0, \qquad u_0 = 1.$$

In this note we shall show the following

THEOREM. *If* $\{p_n; n \geqq 0\}$ *is a sequence of nonnegative measurable functions with* $|Tg| \leqq p_{n+1}$ *whenever* $g \in L^1(m)$ *and* $|g| \leqq p_n$ *then for any* $f \in L^1(m)$,

$$\lim_{n \to \infty} \sum_{k=0}^{n} u_k T^k f(x) \Big/ \sum_{k=0}^{n} u_k p_k(x)$$

exists and is finite a.e. on $\{x \mid \sum_{k=0}^{\infty} u_k p_k(x) > 0\}$.

PROOF. Let I be the positive integers, Σ all possible subsets, and μ the measure on (I, Σ) defined by $\mu(\{1\}) = 1$ and $\mu(\{i\}) = 1 - w_1 - \cdots - w_{i-1}$ for $i \geqq 2$. Let $\{\beta_n; n \geqq 1\}$ be the sequence defined by

$$\beta_n = w_n/(1 - w_1 - \cdots - w_{n-1}),$$

$\beta_1 = w_1$. Let S be the linear operator on $L^1(\mu)$ satisfying $Sh_1 = \sum_{n=1}^{\infty} \beta_n h_n$ and $Sh_i = (1 - \beta_{i-1}) h_{i-1}$ for $i \geqq 2$, where h_n denotes the indicator function of the set $\{n\}$. Then it is known (cf. [2]) that $\|S\| = 1$ and $S^n h_1(1) = u_n$ for each $n \geqq 0$. Thus the direct product $S \times T$ of S and T is a linear contraction on $L^1(\mu \times m)$ and satisfies $(S \times T)^n h_1 f(1, x) = S^n h_1(1) T^n f(x) = u_n T^n f(x)$. Now define a sequence $\{\tilde{p}_n; n \geqq 0\}$ of nonnegative measurable functions on $(I \times X, \Sigma \otimes \mathcal{M}, \mu \times m)$ by $\tilde{p}_n(i, x) = S^n h_1(i) p_n(x)$. It is easily checked that

Received by the editors December 3, 1971.
AMS 1970 *subject classifications*. Primary 47A35.
Key words and phrases. Linear contraction, Chacon's ergodic theorem.

$|(S\times T)\tilde{g}|\leqq\tilde{p}_{n+1}$ whenever $\tilde{g}\in L^1(\mu\times m)$ and $|\tilde{g}|\leqq\tilde{p}_n$. Hence Chacon's ergodic theorem [1] completes the proof of the theorem.

Bibliography

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2. ———, *Ordinary means imply recurrent means*, Bull. Amer. Math. Soc. **70** (1964), 796–797. MR **29** #5977.

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