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A RANDOM TROTTER PRODUCT FORMULA

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ABSTRACT. Let $X(t)$ be a pure jump process with state space S and let $\xi_0, \xi_1, \xi_2, \cdots$ be the succession of states visited by $X(t)$, $\Delta_0\Delta_1 \cdots$ the sojourn times in each state, $N(t)$ the number of transitions before t and $\Delta_t = t - \sum_{k=0}^{N(t)-1} \Delta_k$. For each $x \in S$ let $T_x(t)$ be an operator semigroup on a Banach space L. Define $T_\lambda(t, w) = T_{\xi_0}((1/\lambda)\Delta_0)T_{\xi_1}((1/\lambda)\Delta_1) \cdots T_{\xi_{N(\lambda t)}}((1/\lambda)\Delta_{\lambda t})$. Conditions are given under which $T_\lambda(t, w)$ will converge almost surely (or in probability) to a semigroup of operators as $\lambda \to \infty$. With $S = \{1, 2\}$ and

$$\begin{aligned} X(t) &= 1, \qquad 2n \leqq t < 2n+1, \\ &= 2, \qquad 2n+1 \leqq t < 2n+2, \end{aligned}$$

$n = 0, 1, 2, \cdots$ the result is just the "Trotter product formula".

1. **Introduction.** Let $X(t)$ be a stochastic process with values in a separable, locally compact metric state space S. Of course $X(t)$ is a function from a sample space Ω into S. We will assume that $\Omega = D_S(0, \infty)$, the space of right continuous functions with left hand limits taking values in S and $X(t, w) = w(t)$.

Furthermore we will assume that $X(t)$ is a pure jump process; that is, there is a set $N \subset \Omega$ with $P(N) = 0$ such that for every pair (t, w), $w \notin N$, $X(t, w) = X(t+s, w)$ for all sufficiently small $s > 0$, and $X(t, w)$ has only a finite number of discontinuities in a finite time interval. Under this assumption it makes sense to talk about $\xi_0, \xi_1, \xi_2, \cdots$, the sequence of states visited, and $\Delta_0\, \Delta_1 \cdots$, the sojourn times in these states. In addition we define $N(t)$ to be the number of transitions before time t and

$$\Delta_t = t - \sum_{k=0}^{N(t)-1} \Delta_k.$$

For each $x \in S$ let $T_x(t)$ be a semigroup of linear operators on a Banach space L with infinitesimal operator A_x, satisfying $\|T_x(t)\| \leqq e^{\alpha t}$ for some

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fixed α. We define the random evolution governed by $X(\lambda t)$ by

$$T_\lambda(t, w) = T_{\xi_0}\left(\frac{1}{\lambda}\Delta_0\right) T_{\xi_1}\left(\frac{1}{\lambda}\Delta_1\right) \cdots T_{\xi_{N(\lambda t)}}\left(\frac{1}{\lambda}\Delta_{\lambda t}\right). \tag{1.1}$$

The definition of a random evolution, originally given by Griego and Hersh [**1**] for $X(t)$ a Markov chain, can perhaps best be motivated in the following way:

For each $x \in S$, let $P_x(t, y, \Gamma)$ be a Markov transition function on a measurable state space $(E, \mathscr{B})$, and let $T_x(t)$ be the corresponding semigroup on $B(E, \mathscr{B})$, the space of bounded measurable functions. Let Z be the process (assuming one exists) whose development is governed by $P_x(t, y, \Gamma)$ on time intervals in which X is in state x. Then, at least intuitively,

$$E(f(Z(t)) \mid X(s), s \leqq t, Z(0) = z) = (T_{\xi_0}(\Delta_0) T_{\xi_1}(\Delta_1) \cdots T_{\xi_{N(t)}}(\Delta_t) f)(z).$$

We are interested in the behavior of $T_\lambda(t, w)$ as λ tends to infinity, that is, in what happens if the mode of development of the random evolution (or the process Z) changes at a very rapid rate.

In §2, we will give conditions under which $T_\lambda(t, w)$ converges almost surely to a semigroup whose infinitesimal operator is the closure of $\int A_x f \mu(dx)$ where μ satisfies

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t g(X(s))\, ds = \int g(x) \mu(dx)$$

for all bounded continuous g.

Griego and Hersh [**1**] and Hersh and Pinsky [**2**] consider the case where $\int A_x f \mu(dx) = 0$ (i.e. the limiting semigroup is the identity) and $X(t)$ is a finite Markov chain. They give limit theorems for $E(T_\lambda(\lambda t, w))$ under the assumption that the $T_x(t)$ commute. In a subsequent paper we will show that many of their results hold without the assumption of commutativity.

In what follows we will use a number of different Banach spaces. We will use subscripts on the norm notation only when there is a possibility of confusion (e.g. the norm on L will be denoted by $\|\cdot\|_L$).

2. The limit theorem.

THEOREM (2.1). *Let $X(t)$ be a pure jump process with state space S. Suppose S is a separable, locally compact metric space and there is a measure μ on the Borel subsets of S such that $\mu(S) = 1$ and*

$$P\left\{\lim_{t \to \infty} \frac{1}{t} \int_0^t g(X(s))\, ds = \int g(x) \mu(dx)\right\} = 1 \tag{2.2}$$

for every real, bounded, continuous function g.

For each $x \in S$ let $T_x(t)$ be a semigroup of linear operators on a Banach space L with infinitesimal operator A_x satisfying $\|T_x(t)\| \leqq e^{\alpha t}$, for some α independent of x.

Let D be the set of $f \in L$ such that $A_x f: S \to L$ is a bounded continuous function of x. Define $Af = \int A_x f \mu(dx)$ for $f \in D$.

If D is dense in L and $\mathscr{R}(\mu - A)$ is dense in L for some $\mu > \alpha$, then the closure of A is the infinitesimal operator for a strongly continuous semigroup $T(t)$ defined on L and

$$P\left\{\lim_{\lambda \to \infty} T_\lambda(t, w) f = T(t) f\right\} = 1 \tag{2.3}$$

for every $f \in L$.

To prove Theorem (2.1) we will use the following which is a consequence of the results in [3].

THEOREM (2.4). *For $0 < \lambda < \infty$, let M_λ be a Banach space and $\mathscr{M}$ the Banach space of bounded functions $\lambda \to f(\lambda) \in M_\lambda$ with $\|f(\cdot)\| = \sup_\lambda \|f(\lambda)\|$. Let $\mathrm{LIM}_{\lambda \to \infty}$ denote any notion of limit (e.g. strong convergence, weak convergence) such that $Pf(\cdot) \equiv \mathrm{LIM}_{\lambda \to \infty} f(\lambda)$ defines a bounded linear operator from the subspace of convergent functions into another Banach space M.*

For each λ let $S_\lambda(t)$ be a semigroup of linear operators on M_λ with infinitesimal operator B_λ satisfying $\|S_\lambda(t)\| < e^{\alpha t}$ for some α independent of λ.

Suppose $\mathrm{LIM}_{\lambda \to \infty} f(\lambda) = 0$ implies

$$\mathop{\mathrm{LIM}}_{\lambda \to \infty} S_\lambda(t) f(\lambda) = 0, \quad \text{all } t, \tag{2.5}$$

and

$$\mathop{\mathrm{LIM}}_{\lambda \to \infty} (\mu - B_\lambda)^{-1} f(\lambda) \equiv \mathop{\mathrm{LIM}}_{\lambda \to \infty} \int_0^\infty e^{-\mu t} S_\lambda(t) f(\lambda)\, dt = 0, \quad \text{all } \mu > \alpha. \tag{2.6}$$

Let

$$\mathscr{D}(A) = \left\{ g \in M : \exists f(\cdot) \in \mathscr{M} \ni \mathop{\mathrm{LIM}}_{\lambda \to \infty} f(\lambda) = g \text{ and } \mathop{\mathrm{LIM}}_{\lambda \to \infty} B_\lambda f(\lambda) \equiv Ag \text{ exists} \right\}.$$

(A may be multivalued.)

If $\mathscr{D}(A)$ is dense in M and $\mathscr{R}(\mu - A)$ is dense in M for some $\mu > \alpha$, then the closure of A is the infinitesimal operator of a strongly continuous semigroup $T(t)$ on M and $\mathrm{LIM}_{\lambda \to \infty} f(\lambda) = g \in M$ implies $\mathrm{LIM}_{\lambda \to \infty} S_\lambda(t) f(\lambda) = T(t) g$.

In our application of Theorem (2.4), M_λ will be the space of bounded continuous functions from $D_S(0, \infty)$ into L with $\|g\| = \sup_{w \in D_S} \|g(w)\|$ for all $\lambda > 0$, and M will be L. Let θ_t be the shift operator on $D_S(0, \infty)$, that is $\theta_t w(s)=w(s+t)$. We will say $\mathrm{LIM}_{\lambda\to\infty} f(\lambda, w)=g \in L\equiv M$ if

$$P\left\{\lim_{\lambda\to\infty} \sup_{s\leqq t} \|f(\lambda, \theta_{\lambda s}w) - g\|_L = 0\right\} = 1 \tag{2.7}$$

for all $t>0$. This notion of convergence is stronger than almost sure convergence and weaker than convergence uniform in w. Although we are only interested in almost sure convergence we need the extra strength in order to insure that (2.5) and (2.6) hold.

Finally, the semigroups $S_\lambda(t)$ are given by

$$S_\lambda(t)f(\lambda, w) \equiv T_\lambda(t, w)f(\lambda, \theta_{\lambda t}w). \tag{2.8}$$

If $f(\lambda, w)\equiv f\in L$ we will write $S_\lambda(t)f$.

To complete the proof of Theorem (2.1) we prove the following series of lemmas, all under the assumptions of the theorem.

LEMMA (2.9). *For* $f\in D$,

$$\|T_\lambda(t, w)f - f\|_L \leqq te^{\alpha t} \sup_x \|A_x f\|_L \tag{2.10}$$

and hence, since D is dense in L,

$$\lim_{t\to 0} \sup_{w,\lambda} \|T_\lambda(t, w)g - g\| = 0 \tag{2.11}$$

for all $g\in L$.

PROOF.

$$\begin{aligned}
&\|T_\lambda(t, w)f - f\|_L \\
&\leqq \sum_{k=0}^{N(\lambda t)-1} \left\| T_{\xi_0}\left(\frac{1}{\lambda}\Delta_0\right) \cdots T_{\xi_{k-1}}\left(\frac{1}{\lambda}\Delta_{k-1}\right)\left(T_{\xi_k}\left(\frac{1}{\lambda}\Delta_k\right) - I\right)f\right\|_L \\
&\quad + \left\| T_{\xi_0}\left(\frac{1}{\lambda}\Delta_0\right) \cdots T_{\xi_{N(\lambda t)-1}}\left(\frac{1}{\lambda}\Delta_{N(\lambda t)-1}\right)\left(T_{\xi_{N(\lambda t)}}\left(\frac{1}{\lambda}\Delta_{\lambda t}\right) - I\right)f\right\|_L \\
&\leqq e^{\alpha t}\left(\sum_{k=0}^{N(\lambda t)-1} \frac{1}{\lambda}\Delta_k \|A_{\xi_k}f\|_L + \frac{1}{\lambda}\Delta_{\lambda t}\|A_{\xi_{N(\lambda t)}}f\|_L\right) \\
&\leqq te^{\alpha t}\sup_x \|A_x f\|_L.
\end{aligned} \tag{2.12}$$

LEMMA (2.13). *There is a function* $\varepsilon(\lambda)$ *satisfying* $\lim_{\lambda\to\infty} \varepsilon(\lambda)=0$ *and*

$$P\left\{\lim_{\lambda\to\infty} \sup_{t\leqq T} \left|\frac{1}{\varepsilon(\lambda)}\int_t^{t+\varepsilon(\lambda)} g(X(\lambda s, w))\, ds - \int g(x)\mu(dx)\right| = 0\right\} = 1 \tag{2.14}$$

for every real, bounded continuous function g and every $T>0$.

PROOF. Since the claim is that certain linear functionals of norm one on the space of bounded continuous functions converge to a bounded linear functional of norm one that is given by a measure, it will suffice to prove the result for continuous functions vanishing at infinity. Since the space of continuous functions on S vanishing at infinity is separable, we need only consider a countable dense subset, say $g_1 g_2 g_3 \cdots$.

Note that

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(X(\lambda s, w))\,ds = \frac{1}{\lambda\varepsilon}\int_{\lambda t}^{\lambda(t+\varepsilon)} g(X(s, w))\,ds \tag{2.15}$$

is uniformly continuous in t.

Consequently (2.2) implies

$$P\left\{\lim_{\lambda\to\infty}\sup_{t\leqq T}\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(X(\lambda s, w))\,ds - \int g(x)\mu(dx)\right| = 0\right\} = 1,$$

for every g, $\varepsilon>0$ and $T>0$. Let $\varepsilon_n\to 0$, $T_n\to\infty$ and $\delta_n\to 0$. Then there exists a λ_n such that

$$\sup_{i\leqq n} P\left\{\sup_{\lambda\geqq\lambda_n}\sup_{t\leqq T_n}\left|\frac{1}{\varepsilon_n}\int_t^{t+\varepsilon_n} g_i(X(\lambda s, w))\,ds - \int g(x)\mu(dx)\right| > \delta_n\right\} \leqq \delta_n.$$

The lemma follows by setting $\varepsilon(\lambda)=\varepsilon_n$ for $\lambda_n\leqq\lambda<\lambda_{n+1}$.

LEMMA (2.16). *Let $f\in D$. Then $g(w)=(1/\varepsilon)\int_0^\varepsilon T_\lambda(s, w)f\,ds$ is in $\mathscr{D}(B_\lambda)$ and*

$$B_\lambda g(w) = \frac{1}{\varepsilon}\int_0^\varepsilon T_\lambda(s, w)A_{X(\lambda s, w)}f\,ds. \tag{2.17}$$

PROOF. The fact that $g(w)\in\mathscr{D}(B_\lambda)$ is a standard result of semigroup theory. The form of $B_\lambda g(w)$ is obtained from the following inequality.

$$\begin{aligned} &\left\|\frac{1}{\varepsilon}\int_0^\varepsilon T_\lambda(s, w)A_{X(\lambda s, w)}f\,ds - \frac{S_\lambda(t)g(w) - g(w)}{t}\right\|_L \\ &\quad = \left\|\frac{1}{\varepsilon}\int_0^\varepsilon T_\lambda(s, w)\left(A_{X(\lambda s, w)}f - \frac{T_\lambda(t, \theta_{\lambda s}w)f - f}{t}\right)ds\right\|_L \\ &\quad \leqq \frac{1}{\varepsilon}\int_0^\varepsilon e^{\alpha s}\left\|\left(A_{X(\lambda s, w)}f - \frac{T_\lambda(t, \theta_{\lambda s}w)f - f}{t}\right)\right\|_L ds. \end{aligned} \tag{2.18}$$

Noting that $X(\lambda s, w)=X(0, \theta_{\lambda s}w)$, we observe

$$\left\|A_{X(\lambda s, w)}f - \frac{T_\lambda(t, \theta_{\lambda s}w)f - f}{t}\right\|_L$$

is bounded by (2.10) and goes to zero as t goes to zero for all s. The lemma then follows by the dominated convergence theorem.

LEMMA (2.19). *Let $f \in D$. Define*

$$f(\lambda, w) = \frac{1}{\varepsilon(\lambda)} \int_0^{\varepsilon(\lambda)} T_\lambda(s, w) f \, ds.$$

Then

$$\underset{\lambda \to \infty}{\mathrm{LIM}} f(\lambda, w) = f$$

and

$$\underset{\lambda \to \infty}{\mathrm{LIM}} B_\lambda(\lambda, w) = \int A_x f \mu(dx).$$

PROOF. We must show that

$$\lim_{\lambda \to \infty} \sup_{t \leq T} \left\| B_\lambda f(\lambda, \theta_{\lambda t} w) - \int A_x f \mu(dx) \right\|_L = 0$$

almost surely.

$$\begin{aligned}
&\sup_{t \leq T} \left\| B_\lambda f(\lambda, \theta_{\lambda t} w) - \int A_x f \mu(dx) \right\|_L \\
&\quad = \sup_{t \leq T} \left\| \frac{1}{\varepsilon(\lambda)} \int_0^{\varepsilon(\lambda)} T_\lambda(s, \theta_{\lambda t} w) A_{X(\lambda s, \theta_{\lambda t} w)} f - \int A_x f \mu(dx) \right\|_L \\
(2.20) \qquad &\quad \leqq \sup_{t \leq T} \left\| \frac{1}{\varepsilon(\lambda)} \int_0^{\varepsilon(\lambda)} (T_\lambda(s, \theta_{\lambda t} w) - I) A_{X(\lambda s, \theta_{\lambda t} w)} f \, ds \right\| \\
&\quad\quad + \sup_{t \leq T} \left\| \frac{1}{\varepsilon(\lambda)} \int_0^{\varepsilon(\lambda)} A_{X(\lambda s, \theta_{\lambda t} w)} f \, ds - \int A_x f \mu(dx) \right\|.
\end{aligned}$$

The second term on the right can be rewritten as

$$\sup_{t \leq T} \left\| \frac{1}{\varepsilon(\lambda)} \int_t^{t+\varepsilon(\lambda)} A_{X(\lambda s, w)} f \, ds - \int A_x f \mu(dx) \right\|$$

and goes to zero almost surely by (2.14), the boundedness and continuity of $A_x f$ as a function of x, and the separability and local compactness of S.

The first term on the right of (2.20) can be bounded by

$$\begin{aligned}
&\sup_{t \leq T} \sup_{x \in K} \left\| \frac{1}{\varepsilon(\lambda)} \int_0^{\varepsilon(\lambda)} (T_\lambda(s, \theta_{\lambda t} w) - I) A_x f \right\| \\
(2.21) \qquad &\quad + (2 \sup \|A_x f\|) \left(\sup_{t \leq T} \frac{1}{\varepsilon(\lambda)} \int_0^{\varepsilon(\lambda)} \chi_{K^c}(X(\lambda s, \theta_{\lambda t} w)) \, ds \right).
\end{aligned}$$

A compact set K can be selected so that the lim sup of the second term on the right of (2.21) can be made arbitrarily small. Given a compact K the first goes to zero by the continuity of $A_x f$, the compactness of K and (2.11).

PROOF OF THEOREM (2.1). Lemma (2.19) implies that the operator A in Theorem (2.4) is an extension of $Af = \int A_x f \mu(dx)$. Consequently, under the hypotheses of Theorem (2.1), Theorem (2.4) implies, for all $f \in L$ and $f(w) \equiv f$,

$$\operatorname*{LIM}_{\lambda \to \infty} S_\lambda(t) f(w) = T(t)f.$$

This implies (2.3).

REMARK. Since the probability measure in (2.2) is arbitrary, we have in fact proved convergence for every $w \in D_S(0, \infty)$ that is constant except for a discrete set of jumps and satisfies

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t g(w(t))\, dt = \int g(x) \mu(dx)$$

for all continuous g. Consequently Theorem (2.1) gives a generalization of the "Trotter product formula", that is

THEOREM (2.22) (TROTTER [4]). *Suppose $T(t)$ and $S(t)$ are semigroups of linear operators on L, with infinitesimal operators A and B, satisfying $\|T(t)\| \leqq e^{\alpha t}$ and $\|S(t)\| \leqq e^{\alpha t}$. If $\mathscr{D}(A) \cap \mathscr{D}(B)$ is dense in L and $\mathscr{R}(\mu - \frac{1}{2}(A+B))$ is dense in L for some $\mu > \alpha$ then the closure of $\frac{1}{2}(A+B)$ is the infinitesimal operator of a semigroup $U(t)$ on L and*

$$\lim_{h \to 0} (T(h/2)S(h/2))^{[t/h]} f = U(t)f$$

for all f in L.

We observe that almost sure convergence in (2.2) and (2.3) can be replaced by convergence in probability with only minor alteration in the proof. In particular, the notion of convergence becomes: $\operatorname{LIM}_{\lambda \to \infty} f(\lambda, w) = g \in L$ if

$$\lim_{\lambda \to \infty} P\left\{ \sup_{s \leq t} \|f(\lambda, \theta_{\lambda s} w) - g\| > \varepsilon \right\} = 0$$

for every $\varepsilon > 0$ and every $t > 0$.

EXAMPLE. Let $X(t)$ satisfy the conditions of Theorem (2.1). Let

$$F(x, z) : S \times \boldsymbol{R}^n \to \boldsymbol{R}^n$$

be bounded and satisfy

$$\lim_{x \to x_0} \sup_z |F(x, z) - F(x_0, z)| = 0$$

for all $x_0 \in S$, and

$$\sup_x |F(x, z_1) - F(x, z_2)| < M|z_1 - z_2|$$

for all $z_1, z_2 \in \boldsymbol{R}^n$ and some fixed M. Let $Z_\lambda(t, z)$ be the solution of

$$Z_\lambda(t, z) = z + \int_0^t F(X(\lambda s), Z_\lambda(s, z))\, ds.$$

Theorem (2.1) implies

$$P\left\{\lim_{\lambda\to\infty} \sup_z |Z_\lambda(t, z) - Z(t, z)| = 0\right\} = 1$$

where $Z(t, z)$ is the solution of

$$Z(t, z) = z + \int_0^t \int_S F(x, Z(s, z))\mu(dx)\, ds.$$

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