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METRIZATION OF SYMMETRIC SPACES AND REGULAR MAPS

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ABSTRACT. A symmetric d for a topological space R is said to be *coherent* if whenever $\{x(n)\}$ and $\{y(n)\}$ are sequences in R with $d(x(n), y(n))\to 0$ and $d(x(n), x)\to 0$, then $d(y(n), x)\to 0$. V. Niemytzki and W. A. Wilson have essentially shown that a topological space R is metrizable if and only if R is symmetrizable via a coherent symmetric. Conditions on a symmetric d which are equivalent to d being coherent are established. As a consequence, a theorem of A. Arhangel'skiĭ may be refined by showing that if $f: R\to Y$ is a quotient map from a metrizable space R onto a T_0-space Y, then Y is metrizable if and only if f is a regular map.

For a set R, a nonnegative real valued function d on $R\times R$ is called a *distance function* for R if the following two conditions are satisfied: $d(x, y)=0$ if and only if $x=y$ and $d(x, y)=d(y, x)$. The pair (R, d) is called a *distance space* where R is a set and d is a distance function for R. Two distance functions d and p for a set R are said to be *equivalent* provided that for any sequence $\{x(n)\}$ in R and any point x in R, we have $d(x(n), x)\to 0$ if and only if $p(x(n), x)\to 0$. A distance space (R, d) is *metrizable provided there exists a metric for R which is equivalent to d.* In [7], V. Niemytzki calls a distance function d for a set R *coherent* if whenever $\{x(n)\}$ and $\{y(n)\}$ are sequences in R such that $d(x(n), y(n))\to 0$ and $d(x(n), x)\to 0$, then $d(y(n), x)\to 0$.

THEOREM 1 (NIEMYTZKI, WILSON [7], [9]). *If d is coherent, the distance space (R, d) is metrizable.*

A. H. Frink [5] gives a very elegant proof of Theorem 1. Similar metrization theorems and conditions equivalent to coherence may be found in [4], [5], [7], [9].

Let (R, d) be a distance space. A subset A of R is said to be *d-closed* if and only if $d(x, A)>0$ whenever $x\in R-A$. The complements of d-closed sets form a topology T_d for R. Moreover, if d and p are equivalent

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distance functions for R, then $T_d=T_p$. The triple (R, d, T_d) is called a *symmetric space*. A topological space (R, T) is said to be *symmetrizable* provided there exists a distance function d for R such that $T=T_d$. When speaking of a distance space as a symmetric space, we shall call the distance function a *symmetric*. The concept of a symmetrizable space is due to A. V. Arhangel'skiĭ [3, p. 125]. It is clear from the remarks above that Theorem 1 yields the following: a topological space R is metrizable if and only if R is symmetrizable via a coherent symmetric. Herein lies the importance of the following theorem.

THEOREM 2. *Let d be a symmetric for a topological space R. The following four conditions are equivalent*:

(1) *d is coherent.*

(2) *If $d(x(n), y(n))\to 0$ and $x(n)\to x$, then $y(n)\to x$.*

(3) *If $d(x(n), A)\to 0$ where A is compact, then $\{x(n)\}$ has a subsequence which converges to a point of A.*

(4) *If A and B are disjoint subsets of R, one of which is closed and the other compact, then $d(A, B)>0$.*

PROOF. Let d be coherent, $x\in R$ and $S=\{y\in R: d(x, y)<e\}$ where e is some positive real number. Let $A=R-S$ and $B=\{y\in R: d(y, A)=0\}$.

Let $b\in R$ with $d(b, B)=0$. Then there exists a sequence $\{b(n)\}$ in B with $d(b, b(n))\to 0$. Choose a sequence $\{a(n)\}$ in A with $d(b(n), a(n))\to 0$. Since d is coherent, we have $d(b, a(n))\to 0$ so that $d(b, A)=0$. But then $b\in B$ and it follows that B is a closed set. Consequently, we have $x\in \operatorname{int} S$.

Let $\{x(n)\}$ be a sequence in R and $x\in R$ such that $d(x(n), x)\not\to 0$. Then there exists $e>0$ and a subsequence $\{x(n(i))\}$ of $\{x(n)\}$ such that $x(n(i))\notin S$ for $i=1, 2, \cdots$, where $S=\{y\in R: d(x, y)<e\}$. But, as seen above, $x\in \operatorname{int} S$, so that $x(n(i))\not\to x$, whence $x(n)\not\to x$. By contraposition, we see that if $x(n)\to x$, then $d(x(n), x)\to 0$. That (2) holds is now evident.

Now assume that d satisfies condition (2). Let $\{y(n)\}$ be an arbitrary convergent sequence in R with $y(n)\to y$. Let $Y=\{y\}\cup\{y(1), y(2), \cdots\}$ and let $z\in R-Y$. Suppose that $d(z, Y)=0$. Then there exists a subsequence $\{y(n(i))\}$ of $\{y(n)\}$ with $d(y(n(i)), z)\to 0$. For $i=1, 2, \cdots$, let $z(i)=z$. Then we have $d(y(n(i)), z(i))\to 0$ and $y(n(i))\to y$. Since d satisfies (2), this implies that $z(i)\to y$, which contradicts the fact that R is a T_1-space and $y\neq z$. Therefore, $d(z, Y)>0$, that is, Y is a closed set.

Let B be any nonclosed subset of R. There exists $x\in R-B$ with $d(x, B)=0$. Choose a sequence $\{x(n)\}$ in B with $d(x, x(n))\to 0$ and $x(n)\neq x(m)$ for $n\neq m$. By the previous paragraph, the set $\{x\}\cup\{x(1), x(2), \cdots\}$ is closed in R so that $\{x(1), x(2), \cdots\}$ is closed in the subspace B. Similarly, $\{x(n), x(n+1), \cdots\}$ is closed in B for $n=2, 3, \cdots$. Let $H_n=B-\{x(n), x(n+1), \cdots\}$ for $n=1, 2, \cdots$. Each set H_n is open in the relative topology for B and $\{H_n:n=1,2,\cdots\}$ covers B. Clearly $\{H_n\}$ has no finite subcover so that B is not compact. It follows that compact subsets of R are closed.

We are now in a position to show that d satisfies (3). Let A be a compact subset of R and let $\{x(n)\}$ be a sequence in R with $d(x(n), A)\to 0$. Since A is a closed subset of R, A is itself a symmetrizable space, and therefore sequentially compact. Choose a sequence $\{a(n)\}$ in A with $d(x(n), a(n))\to 0$. There exists a subsequence $\{a(n(i))\}$ of $\{a(n)\}$ and a point $a\in A$ with $a(n(i))\to a$. Since d satisfies (2), it follows that $x(n(i))\to a$, that is, d also satisfies (3).

If d does not satisfy (4), then there exists a compact set A and a closed set B, disjoint from A, with $d(A, B)=0$. Choose a sequence $\{x(n)\}$ in B with $d(x(n), A)\to 0$. Any convergent subsequence of $\{x(n)\}$ must converge to a point of B so that d does not satisfy (3). It follows that if d satisfies (3), then d also satisfies (4).

Finally, let d satisfy condition (4). If A is a compact set and $x\in R-A$, then $d(x, A)>0$ since the singleton set $\{x\}$ is closed and disjoint from A. Consequently, compact subsets of R are closed. Let $\{x(n)\}$ and $\{y(n)\}$ be sequences in R and $x\in R$ such that $d(x(n), y(n))\to 0$ and $d(x, x(n))\to 0$. Suppose that $d(x, y(n))\not\to 0$. Without loss of generality we may suppose that there exists $e>0$ such that $d(x, y(n))>e$ for $n=1, 2, \cdots$. Let $Y=\{y(1), y(2), \cdots\}$ and $X=\{x\}\cup\{x(n):x(n)\notin Y\}$. Since X is compact, disjoint from Y, $d(X, Y)=0$ and d satisfies (4), Y cannot be closed. Let $y\in R-Y$ with $d(y, Y)=0$. Choose a subsequence $\{y(n(i))\}$ of $\{y(n)\}$ such that $d(y, y(n(i)))\to 0$ as $i\to\infty$. Let $Y'=\{y\}\cup\{y(n(i)):i=1, 2, \cdots\}$. Since Y' is compact, it is also closed. Let $X'=X-\{y\}$. Then X' is compact and disjoint from Y' so that $d(X', Y')>0$. But $d(x(n), y(n))\to 0$ implies that $d(X', Y')=0$. The supposition that $d(x, y(n))\not\to 0$ has led to a contradiction; consequently, $d(x, y(n))\to 0$ and d is coherent, completing the proof.

A symmetric d for a topological space R is said to satisfy *condition* A provided that $d(F, K)>0$ whenever F and K are disjoint closed subsets of R, at least one of which is compact. A Hausdorff space is metrizable if and only if it is symmetrizable via a symmetric which satisfies condition A [**3**, Theorem 3.1]. This result also follows immediately from Theorems 1 and 2 above. The following example, due to Peter Harley, shows that there exist symmetrics which satisfy condition A but which are not coherent. Let N be the set of all positive integers. For n, $m\in N$ with $n\neq m$, define $d(n, m)=(|n-m|)^{-1}$. The symmetric d generates the cofinite topology for N, that is, the d-closed sets of N are the finite subsets of N and N itself. The symmetric d clearly satisfies condition A but d is not coherent since (N, d) is not metrizable.

We say that a map $f: R \to Y$ from a metrizable space R onto a space Y is *coherent* if R is symmetrizable by a symmetric d such that if $\{x(n)\}$ and $\{y(n)\}$ are sequences in R with $d(x(n), y(n)) \to 0$ and $f(x(n)) \to y$, then $f(y(n)) \to y$. Every coherent map is continuous. A map $f: R \to Y$ from a metrizable space R onto a topological space Y is said to be *regular* provided there exists a compatible metric p for R such that if $y \in V$ where V is open in Y, then there exists an open neighborhood W of y for which $p(f^{-1}[W], R - f^{-1}[V]) > 0$. Any continuous map from a metrizable space onto a metrizable space is regular [**3**, p. 134]. Every regular map is coherent, as seen in the proof of the following:

LEMMA. *Let $f: R \to Y$ be a function from a metrizable space R onto a metrizable space Y. Then, the following are equivalent*:

(1) *f is continuous.*

(2) *f is regular.*

(3) *f is coherent.*

PROOF. Let p be a metric for R and d be a metric for Y. For $a, b \in R$ define $s(a, b) = p(a, b) + d(f(a), f(b))$. Since f is continuous, the metrics p and s are equivalent. f is regular via s. Now assume that f is a regular map by virtue of a metric p for R. Let $\{x(n)\}$ and $\{y(n)\}$ be sequences in R with $p(x(n), y(n)) \to 0$ and $f(x(n)) \to y$ for some $y \in Y$. Let V be an open set in Y which contains y. Since f is regular by virtue of p, there exists an open neighborhood H of y such that $p(f^{-1}[H], R - f^{-1}[V]) = e > 0$. Since $f(x(n)) \to y$, the sequence $\{x(n)\}$ is eventually in the set $f^{-1}[H]$. Since $p(x(n), y(n)) \to 0$, it follows that the sequence $\{y(n)\}$ must eventually be in the set $f^{-1}[V]$, that is $\{f(y(n))\}$ is eventually in V so that $f(y(n)) \to y$, proving that f is a coherent map. That (3) implies (1) is almost immediate, completing the proof.

Let $f: R \to Y$ be a quotient map from a metrizable space R onto a T_1-space Y. In [**3**, p. 134], Arhangel'skiĭ established the following results: Y is metrizable if and only if f is regular and pseudo-open; and, if Y is Hausdorff, then Y is metrizable if and only if f is regular. The following sharpens these results and completes the solution to a problem raised in [**1**, p. 368].

THEOREM 3. *Let $f: R \to Y$ be a quotient map from a metrizable space R onto a T_0-space Y. Then, the following are equivalent*:

(1) *Y is metrizable.*

(2) *f is a regular map.*

(3) *f is a coherent map.*

PROOF. That (1) implies (2) and (2) implies (3) follows from the Lemma. Therefore, assume that f is a coherent map by virtue of a symmetric p for R. Define a function d on $Y \times Y$ by $d(x, y) = p(f^{-1}(x), f^{-1}(y))$.

Since Y is T_0 and f is a coherent map via p, we have $d(x, y)>0$ for $x\neq y$, that is, d is a distance function. Let Q denote the quotient topology for Y relative to f and T_d denote the topology consisting of the complements of d-closed sets. Since f is continuous with respect to T_d, we have $T_d\subset Q$. Let A be closed in the space (Y, Q) and let $x\in Y-A$. If $d(x, A)=0$, then there exist sequences $\{x(n)\}$ in $f^{-1}(x)$ and $\{a(n)\}$ in $f^{-1}[A]$ with $p(a(n), x(n))\to 0$. But then $f(a(n))\to x$ in (Y, Q) since f is coherent, contradicting the assumption that A is closed in (Y, Q). We must have $d(x, A)>0$, that is, A is closed in (Y, T_d). It follows that $Q=T_d$, that is, Y is symmetrizable via the symmetric d.

Let $\{x(n)\}$ and $\{y(n)\}$ be sequences in Y and $x\in Y$ such that $d(x(n), y(n))\to 0$ and $x(n)\to x$. There exist sequences $\{x'(n)\}$ and $\{y'(n)\}$ in R with $p(x'(n), y'(n))\to 0$ where $f(x'(n))=x(n)$ and $f(y'(n))=y(n)$. Then $f(x'(n))\to x$ and since f is a coherent map via p, $f(y'(n))\to x$, that is, $y(n)\to x$. It follows from Theorem 2 that d is a coherent symmetric. The metrizability of Y now follows by Theorem 1, completing the proof.

A map $f: R\to Y$ is said to be *proper* if $f^{-1}[A]$ is compact whenever A is compact in Y. The following is easy to verify: Let Y be a topological space in which compact subsets are closed; if $f: R\to Y$ is a proper map from a metrizable space R onto Y, then f is a coherent map. As an immediate consequence we have that the proper quotient image of a metric space is metrizable. In fact, if $f: R\to Y$ is a proper quotient map from a metrizable space R onto a space Y, then since Y is metrizable, by a theorem of G. T. Whyburn [**10**] or by Theorem 2.15 of [**2**], f is actually a closed map. In short, proper quotient maps on metric spaces are always perfect maps. Therefore, Theorem 3 yields an easy proof of the well-known Morita-Hanai-Stone Theorem [**6**], [**8**] that the perfect image of a metric space is metrizable.

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