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A GENERALIZATION OF TWO INEQUALITIES INVOLVING MEANS[1]

SCOTT LAWRENCE AND DANIEL SEGALMAN

ABSTRACT. Fan has proven an inequality relating the arithmetic and geometric means of $(x_1, \cdots, x_n)$ and $(1-x_1, \cdots, 1-x_n)$, where $0<x_i \leqq \frac{1}{2}$, $i=1, \cdots, n$. Levinson has generalized Fan's inequality; his result involves functions with positive third derivatives on $(0, 1)$. In this paper, the above condition that requires $0<x_i \leqq \frac{1}{2}$ has been replaced by a condition which only weights the x_i to the left side of $(0, 1)$ in pairs, and Levinson's differentiability requirement has been replaced by the analogous condition on third differences.

1. **Introduction.** Levinson [1] has generalized the following inequality of Ky Fan:

Let $0<x_i \leqq \frac{1}{2}$, $i=1, \cdots, n$. Then

$$\prod_{i=1}^{n} x_i \Big/ \left(\sum_{i=1}^{n} x_i\right)^n \leqq \prod_{i=1}^{n} (1-x_i) \Big/ \left(\sum_{i=1}^{n} (1-x_i)\right)^n$$

with equality if and only if all the x_i are equal.

Levinson's result is the following:

Let f have a third derivative on $(0, 2a)$ with $f'''(x) \geqq 0$ for all x in $(0, 2a)$. If $0<x_i \leqq a$ and $0<p_i$, $i=1, \cdots, n$, then

$$\begin{aligned}(1)\quad &\sum_{i=1}^{n} p_i f(x_i) \Big/ \sum_{i=1}^{n} p_i - f\left(\sum_{i=1}^{n} p_i x_i \Big/ \sum_{i=1}^{n} p_i\right) \\ &\leqq \sum_{i=1}^{n} p_i f(2a-x_i) \Big/ \sum_{i=1}^{n} p_i - f\left(\sum_{i=1}^{n} p_i(2a-x_i) \Big/ \sum_{i=1}^{n} p_i\right).\end{aligned}$$

Furthermore, if $f'''(x)>0$ on $(0, 2a)$, then the above inequality reduces to equality if and only if all the x_i are equal.

Ky Fan's inequality is Levinson's inequality with $p_i=1$, $a=\frac{1}{2}$, and $f(x)=\log x$. Popoviciu [2] has slightly weakened the differentiability assumption in Levinson's theorem.

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Our purpose is to prove Levinson's inequality and certain special cases of it under more natural, less restrictive conditions than the above.

2. **Main result.** The following theorem extends Levinson's inequality in the case of equal weights.

THEOREM 1. *Let f be a continuous function defined on $(0, 2a)$ for which $\Delta_h^3 f(x)>0$ for all x in $(0, 2a)$ and $h>0$ for which $\Delta_h^3 f(x)$ is defined (i.e., for all x in $(0, 2a)$ and $h>0$ for which $x+3h<2a$). Let $x_1, \cdots, x_n$ be numbers in $(0, 2a)$ such that $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ and $x_i+x_{n+1-i} \leqq 2a$, $i=1, \cdots, n$. Then*

$$(2)\quad \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leqq \frac{1}{n}\sum_{i=1}^{n} f(2a - x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} (2a - x_i)\right)$$

with equality if and only if either all the x_i are equal or $x_i+x_{n+1-i}=2a$, $i=1, \cdots, n$.

Several remarks are in order concerning the statement of the theorem. First of all, the order of the x_i is obviously irrelevant. We merely need to be able to pair them in such a manner that the sum of the members of each pair is at most $2a$. The ordering condition in the theorem is motivated entirely by notational convenience. Secondly, if f is continuous and $\Delta_h^3 f(x)>0$ for all x and $h>0$, then $\Delta_j\Delta_k\Delta_l f(x)>0$ for all x and $j, k, l>0$ (a direct proof is not immediate, but is fairly easy). In the proof of the theorem we shall only need that $\Delta_k^2\Delta_h f(x)>0$. Of course, if f is three times differentiable and $f'''(x)>0$ for all x in $(0, 2a)$, then third differences are also positive. Conversely, if third differences are positive, then $f'''(x)$ is nonnegative (but not necessarily strictly positive). The third and probably most important feature of the theorem different from Levinson's theorem is that the x_i are somewhat less restricted. Finally, if we retain Levinson's restriction on the x_i, i.e. $0<x_i \leqq a$, then the theorem for arbitrary weights can be proven as an easy corollary to the theorem for equal weights. Theorem 1 can be proven under the assumption that $\Delta_h^3 f(x) \geqq 0$ as a limiting case of the theorem as stated above, but the cases of equality are then obscured.

PROOF OF THE THEOREM. If all the x_i are equal, both sides of the inequality reduce to zero. If $x_i+x_{n+1-i}=2a$ for all i, then the right side is the same as the left except for a reversal in the order of the summations.

Suppose not all the x_i are equal and that they are not summable in pairs of $2a$. Furthermore, in the pairing hypothesis assume first that some pair (let it be (x_1, x_n) for convenience) has sum less than $2a$ and has unequal members. Then let $h=2a-x_n-x_1$, $k=(x_n-x_1)/2$. Since $x_1+x_n<2a$

and $x_1 < x_n$, we have $h > 0$, $k > 0$. Thus,

$$\begin{aligned} 0 &< \Delta_k^2 \Delta_h f(x_1) \\ &= f(x_1 + h + 2k) - 2f(x_1 + h + k) \\ &\quad + f(x_1 + h) - f(x_1 + 2k) + 2f(x_1 + k) - f(x_1) \\ &= f(2a - x_1) - 2f(2a - (x_1 + x_n)/2) + f(2a - x_n) \\ &\quad - f(x_n) + 2f((x_1 + x_n)/2) - f(x_1). \end{aligned}$$

Hence,

$$\begin{aligned} &2f((x_1 + x_n)/2) - 2f(2a - (x_1 + x_n)/2) \\ &\qquad > f(x_1) + f(x_n) - f(2a - x_1) - f(2a - x_n). \end{aligned}$$

Thus we have proven that replacing each of x_1 and x_n by $(x_1+x_n)/2$ increases the function

$$l(x_1, \cdots, x_n) = \frac{1}{n}\left(\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(2a - x_i)\right).$$

Given any n-tuple of real numbers $(c_1, \cdots, c_n)$ subject to the conditions $0 < c_i < 2a$, $c_i + c_{n+1-i} \leqq 2a$, $i = 1, \cdots, n$, and $c_1 \leqq c_2 \leqq \cdots \leqq c_n$, we consider the compact set $D \subseteq \boldsymbol{R}^n$ defined as follows: $(x_1, \cdots, x_n)$ is in D if and only if

$$\begin{gathered} c_1 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n \leqq c_n; \\ x_1 + x_2 + \cdots + x_n = c_1 + c_2 + \cdots + c_n; \\ x_i + x_{n+1-i} \leqq 2a, \qquad i = 1, \cdots, n. \end{gathered}$$

Since $l(x_1, \cdots, x_n)$ is continuous on D, it takes on a maximum value there. Let $l(y_1, \cdots, y_n)$ be this maximum for some $(y_1, \cdots, y_n)$ in D. We have that either $y_1 = y_2 = \cdots = y_n$ or $y_i + y_{n+1-i} = 2a$, $i = 1, \cdots, n$, as follows. If there is some j for which $y_j \neq y_{n+1-j}$ and $y_j + y_{n+1-j} < 2a$, then replacing y_j and y_{n+1-j} by $(y_j + y_{n+1-j})/2$ and renumbering, we would obtain an n-tuple (in D) $(z_1, \cdots, z_n)$ for which $l(z_1, \cdots, z_n) > l(y_1, \cdots, y_n)$, as above. Hence, there are no maximizing n-tuples with these properties, so for each i either $y_i = y_{n+1-i}$ or $y_i + y_{n+1-i} = 2a$.

Suppose this is the case. We know that

$$r(x_1, \cdots, x_n) = f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - f\left(\frac{1}{n}\sum_{i=1}^{n}(2a - x_i)\right)$$

remains constant on D. We shall show that unless all the pairs of the x_i sum to $2a$ or all the x_i are equal, then $l(x_1, \cdots, x_n) < r(x_1, \cdots, x_n)$. Since equality holds when $y_1 = y_2 = \cdots = y_n = (c_1 + \cdots + c_n)/n$ $((y_1, \cdots, y_n)$

is in D in this case), we will have eliminated all but the stated possible maximizing n-tuples.

Suppose that for some j we have that $y_j = y_{j+1} = \cdots = y_{n+1-j}$ (if $y_j = y_{n+1-j}$, this follows from the ordering), that $y_j < a$, and that $y_i + y_{n+1-i} = 2a$ for all $i < j$. Then

$$\frac{1}{n}\sum_{i=1}^{n} f(y_i) - \frac{1}{n}\sum_{i=1}^{n} f(2a - y_i)$$
$$= \frac{1}{n}\sum_{i=j}^{n+1-j} (f(y_i) - f(2a - y_i)) = \frac{n+2-2j}{n}[f(y_j) - f(2a - y_j)],$$

and

$$f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) - f\left(\frac{1}{n}\sum_{i=1}^{n}(2a - y_i)\right) = f\left(\frac{2j-2}{n}a + \frac{n+2-2j}{n}y_j\right)$$
$$- f\left(\frac{n+2-2j}{n}(2a - y_j) + \frac{2j-2}{n}a\right).$$

Let $\omega = (n+2-2j)/n$, $\delta = a - y_j$. Consider

$$\frac{1}{n}\sum_{i=1}^{n} f(2a - y_i) - \frac{1}{n}\sum_{i=1}^{n} f(y_i) + f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) - f\left(\frac{1}{n}\sum_{i=1}^{n}(2a - y_i)\right)$$
$$= \frac{n+2-2j}{n}[f(2a - y_j) - f(y_j)] + f\left(\frac{2j-2}{n}a + \frac{n+2-2j}{n}y_j\right)$$
$$- f\left(\frac{n+2-2j}{n}(2a - y_j) + \frac{2j-2}{n}a\right)$$
$$= \omega[f(a+\delta) - f(a-\delta)] + [f(a - \omega\delta) - f(a + \omega\delta)] = G(\omega).$$

We would like to show that $G(\omega) > 0$ unless either all of the y_i are equal ($\omega = 1$), or $y_i + y_{n+1-i} = 2a$, $i = 1, \cdots, n$ ($\omega = 0$). We know that $G(0) = 0 = G(1)$. Let c and d be any strictly positive numbers. Then a simple observation reveals that $\Delta_c\Delta_d G(\omega) = -\Delta_c\Delta_d\Delta_{2\omega\delta} f(a - \omega\delta) < 0$ if ω is in $(0, 1)$ and either difference is defined. Hence, $G(\omega)$ is a strictly concave function on $(0, 1)$ with $G(0) = 0 = G(1)$, so $G(\omega) > 0$ for ω in $(0, 1)$. This proves that the only cases of equality are those stated above, so the proof of the theorem is complete.

3. **Integral generalization.** By similar methods we obtain the following generalization.

THEOREM 2. *Let f be a continuous function on $(0, 2a)$ for which $\Delta_h^3 f(x) > 0$ for all x in $(0, 2a)$ and $h > 0$, and let g be an integrable positive* *function on* (0, 1) *such that* $g(t)+g(1-t)\leqq 2a$ *for all* t *in* (0, 1). *Then*

$$\int_0^1 f(g(t))\,dt - f\left(\int_0^1 g(t)\,dt\right) \leqq \int_0^1 f(2a-g(t))\,dt - f\left(\int_0^1 (2a-g(t))\,dt\right) \tag{3}$$

with equality if and only if either $g(t)$ *is constant almost everywhere on* (0, 1) *or* $g(t)+g(1-t)=2a$ *almost everywhere on* (0, 1).

This theorem can be specialized to a generalized version of (2) as follows. Let $\{p_i\}$ and $\{x_i\}$ be sequences of n positive real numbers with $x_i<2a$, $i=1,\cdots,n$. Define $p_0=0$, $g(t)=x_j$ for $\sum_{i=0}^{j-1} p_i/\sum_{i=0}^{n} p_i<t\leqq \sum_{i=0}^{j} p_i/\sum_{i=0}^{n} p_i$, and suppose that $\{p_i\}$ and $\{x_i\}$ are such that $g(t)+g(1-t)\leqq 2a$ for all t in (0, 1). Then, inequality (3) yields

$$\sum_{i=1}^n p_i f(x_i)\Big/\sum_{i=1}^n p_i - f\left(\sum_{i=1}^n p_i x_i\Big/\sum_{i=1}^n p_i\right) \leqq \sum_{i=1}^n p_i f(2a-x_i)\Big/\sum_{i=1}^n p_i - f\left(\sum_{i=1}^n p_i(2a-x_i)\Big/\sum_{i=1}^n p_i\right), \tag{4}$$

with equality if and only if either all the x_i are equal or $g(t)+g(1-t)=2a$ for all but a finite set of t.

Inequality (4) essentially generalizes (1) to the case where the means are weighted to the left. If we retain Levinson's restriction on the $\{x_i\}$, (4) yields (1).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706