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EXISTENCE OF SOLUTIONS OF ABSTRACT MEASURE DIFFERENTIAL EQUATIONS

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ABSTRACT. In an earlier paper the author has introduced an abstract measure differential equation as a generalization of ordinary differential equations and measure differential equations, and proved a theorem for the existence and uniqueness of solutions of this equation. In the present paper the problem of the existence of solutions is further investigated.

1. **Introduction.** An abstract measure differential equation is defined in [3] as follows. Let X be a linear space over the field $\mathscr{F}$ where $\mathscr{F}$ is the set $\boldsymbol{R}$ of real numbers or the set $\boldsymbol{C}$ of complex numbers. For each $x \in X$, define

$$S_x = \{\alpha x: -\infty < \alpha < 1\}, \qquad \bar{S}_x = \{\alpha x: -\infty < \alpha \leqq 1\} \quad \text{if } \mathscr{F} = \boldsymbol{R};$$
$$S_x = \{\alpha x: 0 \leqq |\alpha| < 1\}, \qquad \bar{S}_x = \{\alpha x: 0 \leqq |\alpha| \leqq 1\} \quad \text{if } \mathscr{F} = \boldsymbol{C}.$$

Let $S \subset X$ where if S is a proper subset of X, it is of the form S_ξ for some $\xi \in X$. Let $\mathscr{M}$ be a σ-algebra in S containing the sets $\bar{S}_x$ for all $x \in S$. We shall denote by $\mathrm{ca}(S, \mathscr{M})$ the space of all countably additive scalar functions (i.e. real measures or complex measures) on $\mathscr{M}$. Let $\Omega \subset \mathscr{F}$ be defined by

$$\Omega = \{\alpha : |\alpha| < a\}.$$

Abstract measure differential equations are equations of the form

$$(*) \qquad d\lambda/d\mu = f(x, \lambda(\bar{S}_x))$$

where μ is a positive σ-finite measure or a complex measure on $\mathscr{M}$, $d\lambda/d\mu$ is Radon-Nikodym derivative of a measure $\lambda \in \mathrm{ca}(S, \mathscr{M})$ with respect to μ, and f is a function defined on $S \times \Omega$ such that $f(x, \lambda(\bar{S}_x))$ is μ-integrable on S for each $\lambda \in \mathrm{ca}(S, \mathscr{M})$.

The solution of $(*)$ is defined as follows. Let $\alpha_0 \in \Omega$, $x_0 \in S$, $\bar{S}_{x_0} \subset X_0 \in \mathscr{M}$ and let $\mathscr{M}_0$ be the smallest σ-algebra in X_0 containing $\bar{S}_{x_0} - S_{x_0}$ and the sets $\bar{S}_x$ for $x \in X_0 - S_{x_0}$. A measure $\lambda \in \mathrm{ca}(X_0, \mathscr{M}_0)$ is called a solution of $(*)$

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on X_0 with initial data $[\bar{S}_{x_0}, \alpha_0]$ (to be denoted by $\lambda[X_0; \bar{S}_{x_0}, \alpha_0]$) if $\lambda(\bar{S}_{x_0})=\alpha_0$, $\lambda(E) \in \Omega$ for $E \in \mathscr{M}_0$, $\lambda \ll \mu$ on $X_0 - S_{x_0}$ and λ satisfies (*) a.e. $[\mu]$ on $X_0 - S_{x_0}$.

An existence and uniqueness theorem is proved in [3] where one of the assumptions for f is to satisfy a Lipschitz condition in α. In the present paper the existence of a solution (without claiming its uniqueness) is established by changing some of the assumptions, particularly the Lipschitz condition hypothesis for f is replaced by the continuity of f in α for fixed x. Taking $\mathrm{ca}(X, \mathscr{M})$ to be the space of real measures, maximum and minimum solutions are defined and their existence is demonstrated under the hypotheses of the existence theorem. It may be noted that Theorem 1 of this paper includes and extends [**1**, Theorem 1.1, p. 43] and Theorem 2 includes and extends [**1**, Theorem 1.2, p. 45].

2. **Existence of solutions.** For any $E_1, E_2 \in \mathscr{M}$, we define

$$\rho(E_1, E_2) = |\mu|\,(E_1 - E_2) + |\mu|\,(E_2 - E_1) \tag{2.1}$$

where $|\mu|$ denotes the total variation measure of μ. It is clear that for any $E_1, E_2 \in \mathscr{M}$,

$$\rho(E_1, E_2) \geqq 0, \qquad \rho(E_1, E_2) = 0 \quad \text{if } E_1 = E_2,$$
$$\rho(E_1, E_2) = \rho(E_2, E_1).$$

Also, for any $E_1, E_2, E_3 \in \mathscr{M}$, we have

$$\rho(E_1, E_2) \leqq \rho(E_1, E_3) + \rho(E_3, E_2),$$

which follows from

$$(E_1 - E_2) \subset (E_1 - E_3) \cup (E_3 - E_2),$$
$$(E_2 - E_1) \subset (E_3 - E_1) \cup (E_2 - E_3).$$

The function ρ thus defines a pseudometric for $\mathscr{M}$.

We shall now prove the following existence theorem.

THEOREM 1. *Let* $\alpha_0 \in \Omega$, $x_0 \in S$ *and let for each* $\xi \in S - \bar{S}_{x_0}$ *the smallest* σ-*algebra containing* $\bar{S}_{x_0} - S_{x_0}$ *and* $\bar{S}_x$, $x \in \bar{S}_\xi - S_{x_0}$, *be compact in the topology generated by the pseudometric* ρ *defined by* (2.1). *Then there exists a solution* $\lambda_0 = \lambda_0[\bar{S}_{x'}; \bar{S}_{x_0}, \alpha_0]$ *of* (*) *for some* $x' \in S - \bar{S}_{x_0}$ *if the following conditions are satisfied*:

(i) $\mu(S_{x_0}) \neq 0$ *and* $|\mu|(\bar{S}_x - S_x) = 0$ *for each* $x \in S - S_{x_0}$;

(ii) *there exists a* μ-*integrable function* w *on* S *such that* $|f(x, \alpha)| \leqq w(x)$ *uniformly in* $\alpha \in \Omega$;

(iii) f *is continuous in* α *for each fixed* x.

PROOF. Choose a real number $r>1$ such that

$$\int_{\bar{S}_{rx_0}-S_{x_0}} w(x)\, d\,|\mu| < a - |\alpha_0|. \tag{2.2}$$

It is possible to choose such a real number r (see the proof of Theorem 1 in [3]).

Let $\mathscr{M}_0$ be the smallest σ-algebra containing $\bar{S}_{x_0}-S_{x_0}$ and all the sets of the form $\bar{S}_x$ for $x\in\bar{S}_{rx_0}-S_{x_0}$. In what follows, it will always be assumed that $E\in\mathscr{M}_0$. Define measures λ_j by

$$\begin{aligned} \lambda_j(E) &= \alpha_0 && \text{for } E=\bar{S}_{x_0},\\ &=0 && \text{for } E\subset \bar{S}_{x_0+\xi_j}-S_{x_0},\\ &=\int_E f(x,\lambda_j(\bar{S}_{x-\xi_j}))\,d\mu && \text{for } E\subset \bar{S}_{rx_0}-S_{x_0+\xi_j},\end{aligned} \tag{2.3}$$

$$j=1,2,\cdots,$$

where

$$\xi_j = [(r-1)/j]x_0. \tag{2.4}$$

Then λ_1 is defined by the first two expressions in (2.3). For any fixed $j\geqq 2$, the first and second expressions in (2.3) define λ_j for $E\subset\bar{S}_{x_0+\xi_j}$; and since $(x,0)$ and $(x,\alpha_0)\in\bar{S}_{rx_0}\times\Omega$, the last expression defines λ_j for $E\subset\bar{S}_{x_0+2\xi_j}-S_{x_0+\xi_j}$. λ_j is then defined for $E\subset\bar{S}_{x_0+2\xi_j}$. Also, for $E\subset\bar{S}_{x_0+2\xi_j}$, we have

$$|\lambda_j(E)| \leqq |\alpha_0| + \int_{\bar{S}_{rx_0}-S_{x_0}} w(x)\,d\,|\mu| < a, \tag{2.5}$$

by condition (ii) and (2.2); and, therefore,

$$\lambda_j(E)\in\Omega. \tag{2.6}$$

Assume that λ_j is defined for $E\subset\bar{S}_{x_0+k\xi_j}$ when $2\leqq k<j$. Then the last expression in (2.3) defines λ_j for $\bar{S}_{x_0+(k+1)\xi_j}-S_{x_0+k\xi_j}$ and thus λ_j is defined for all $E\subset\bar{S}_{x_0+(k+1)\xi_j}$. Also, for such E's, $\lambda_j(E)$ satisfies (2.5) and hence (2.6) because of condition (ii) and (2.2). Therefore, by induction, (2.3) defines λ_j on $\mathscr{M}_0$.

Let $E_1, E_2\in\mathscr{M}_0$ be such that $\rho(E_1,E_2)<|\mu|(S_{x_0})$. Then $\bar{S}_{x_0}$ is a subset either of both E_1 and E_2 or of neither of E_1 and E_2. For, otherwise,

$$\rho(E_1,E_2) = |\mu|\,(E_1-E_2) + |\mu|\,(E_2-E_1) \geqq |\mu|\,(S_{x_0}).$$

It can be verified that either

$$\lambda_j(E_1)-\lambda_j(E_2)=0$$

or else

$$\lambda_j(E_1) - \lambda_j(E_2) = \int_{E_1-E_2} f(x, \lambda_j(\bar{S}_{x-\xi_j}))\, d\mu - \int_{E_2-E_1} f(x, \lambda_j(\bar{S}_{x-\xi_j}))\, d\mu,$$

and therefore, by condition (ii),

$$\text{(2.7)}\quad \begin{aligned} |\lambda_j(E_1) - \lambda_j(E_2)| &< \int_{E_1-E_2} w(x)\, d\,|\mu| + \int_{E_2-E_1} w(x)\, d\,|\mu| \\ &= \int_{(E_1-E_2)\cup(E_2-E_1)} w(x)\, d\,|\mu|. \end{aligned}$$

By Dunford and Schwartz [**2**, Theorem 20, p. 114], if

$$\text{(2.8)}\qquad \nu(E) = \int_E w(x)\, d\,|\mu|$$

then

$$\lim_{|\mu|(E)\to 0} \nu(E) = 0.$$

This implies that the function ν defined by (2.8) is continuous (and hence uniformly continuous) on the compact pseudometric space $\mathscr{M}_0$. From the uniform continuity of ν on $\mathscr{M}_0$, it follows from (2.7) that given any $\varepsilon>0$, there exists a $\delta(\varepsilon)$, $0<\delta$ $(<|\mu|(S_{x_0}))$ such that $|\lambda_j(E_1)-\lambda_j(E_2)|<\varepsilon$ whenever $|\mu|(E_1-E_2)+|\mu|(E_2-E_1)=\rho(E_1,E_2)<\delta$; i.e. $\{\lambda_j\}$ is an equicontinuous set. Moreover, $\{\lambda_j\}$ is uniformly bounded since, by (2.5), we have

$$\text{(2.9)}\qquad \sup_{E\in\mathscr{M}_0} |\lambda_j(E)| \leqq |\alpha_0| + \int_{\bar{S}_{rx_0}-S_{x_0}} w(x)\, d\,|\mu| < a.$$

Therefore, it follows by the Arzela-Ascoli theorem [**2**, p. 266] that $\{\lambda_j\}$ is conditionally compact in the space $C(\mathscr{M}_0)$ of all bounded continuous scalar functions on $\mathscr{M}_0$. Hence there exists a subsequence $\{\lambda_{j_k}\}$ of $\{\lambda_j\}$ and a function $\lambda_0 \in C(\mathscr{M}_0)$ such that $\lambda_{j_k}\to\lambda_0$ uniformly on $\mathscr{M}_0$ as $k\to\infty$. And since

$$\rho(\bar{S}_{x-\xi_{j_k}}, \bar{S}_x) = |\mu|\,(\bar{S}_x - \bar{S}_{x-\xi_{j_k}}) \to |\mu|\,(\bar{S}_x - S_x) = 0,$$

by condition (i) it follows that $\lambda_{j_k}(\bar{S}_{x-\xi_{j_k}})\to\lambda_0(\bar{S}_x)$. Also, by (2.9), $\lambda_0(E)\in\Omega$ for $E\in\mathscr{M}_0$. Hence the continuity of f in α for fixed x (condition (iii)) implies

$$\text{(2.10)}\qquad \lim_{k\to\infty} f(x, \lambda_{j_k}(\bar{S}_{x-\xi_{j_k}})) = f(x, \lambda_0(\bar{S}_x)).$$

From (2.10) and condition (ii), it follows by Lebesgue's dominated convergence theorem that

$$\lim_{k\to\infty} \int_E f(x, \lambda_{j_k}(\bar{S}_{x-\xi_{j_k}}))\, d\mu = \int_E f(x, \lambda_0(\bar{S}_x))\, d\mu.$$

Now replacing j by j_k in (2.3) and letting $k\to\infty$, we obtain

$$\text{(2.11)}\qquad \begin{aligned} \lambda_0(E) &= \alpha_0 && \text{for } E = \bar{S}_{x_0}, \\ &= \int_E f(x, \lambda_0(\bar{S}_x))\, d\mu && \text{for } E \subset \bar{S}_{rx_0} - S_{x_0}. \end{aligned}$$

Thus λ_0 satisfies $(*)$ a.e. $[\mu]$ on $\bar{S}_{rx_0}-S_{x_0}$. It also follows from (2.11) that λ_0 is countably additive on $\mathcal{M}_0$. Hence λ_0 is a solution of $(*)$ on $\bar{S}_{rx_0}$ with initial condition $\lambda_0(\bar{S}_{x_0})=\alpha_0$. This completes the proof.

3. **Maximum and minimum solutions.** In this section we shall be concerned with real measures only and so $\mathrm{ca}(S,\mathcal{M})$ will now denote the space of real measures on $\mathcal{M}$. Let α_0, x_0, X_0 and $\mathcal{M}_0$ be the same as in §1. Let Λ be the set of all solutions of $(*)$ on X_0 with initial data $[\bar{S}_{x_0}, \alpha_0]$.

DEFINITION. A maximum solution of $(*)$ on X_0 with initial data $[\bar{S}_{x_0}, \alpha_0]$ is a solution $\lambda_M \in \Lambda$ with the property

$$\lambda(E) \leqq \lambda_M(E) \qquad (E \in \mathcal{M}_0)$$

for each $\lambda \in \Lambda$. Similarly, $\lambda_m \in \Lambda$ will be called a minimum solution of $(*)$ on X_0 with initial data $[S_{x_0}, \alpha_0]$ if

$$\lambda(E) \geqq \lambda_m(E) \qquad (E \in \mathcal{M}_0)$$

for each $\lambda \in \Lambda$.

We shall now establish the existence of λ_M and λ_m under the assumptions of Theorem 1.

THEOREM 2. *Let the hypotheses of Theorem 1 be satisfied. Then there exist a maximum solution λ_M and a minimum solution λ_m of* $(*)$ *on $X_0=\bar{S}_{rx_0}$ ($\bar{S}_{rx_0}$ being defined by* (2.2)) *with initial data* $[\bar{S}_{x_0}, \alpha_0]$.

PROOF. Define $\lambda^*(E)=\sup_{\lambda\in\Lambda}\{\lambda(E)\}$ $(E\in\mathcal{M}_0)$. We shall prove that λ^* is a maximal solution λ_M.

Clearly $\lambda^*(\bar{S}_{x_0})=\alpha_0$ and $\lambda^*\ll\mu$ on $X_0-S_{x_0}$. By condition (ii) of Theorem 1, each $\lambda\in\Lambda$ satisfies the inequality

$$\text{(3.1)}\qquad |\lambda(E)| < |\alpha_0| + \int_{\bar{S}_{rx_0}-S_{x_0}} w(x)\, d\,|\mu| \qquad (E \in \mathcal{M}_0).$$

And, since

$$|\lambda^*(E)| = \left|\sup_{\lambda\in\Lambda} \lambda(E)\right| \leqq \sup_{\lambda\in\Lambda} |\lambda(E)| \qquad (E \in \mathcal{M}_0).$$

it follows from (3.1) and (2.2) that $\lambda^*(E)\in\Omega$ for each $E\in\mathcal{M}_0$.

We shall now show that λ^* satisfies $(*)$ a.e. $[\mu]$ on $X_0-S_{x_0}$. It can be shown, in the same way as the equicontinuity of $\{\lambda_j\}$ was demonstrated in the proof of Theorem 1, that Λ is an equicontinuous set. Thus, given any

$\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)>0$ such that $E_1, E_2\in\mathcal{M}_0$ and $\rho(E_1, E_2)<\delta$ imply

$$(3.2)\qquad |\lambda(E_1)-\lambda(E_2)|<\varepsilon \qquad \text{for each } \lambda\in\Lambda,$$

and hence

$$(3.3)\qquad \begin{aligned} |\lambda^*(E_1)-\lambda^*(E_2)| &= \left|\sup_{\lambda\in\Lambda}\lambda(E_1)-\sup_{\lambda\in\Lambda}(E_2)\right| \\ &\leqq \left|\sup_{\lambda\in\Lambda}(\lambda(E_1)-\lambda(E_2))\right| \\ &\leqq \sup_{\lambda\in\Lambda}|\lambda(E_1)-\lambda(E_2)|\leqq\varepsilon. \end{aligned}$$

Let $x_1, x_2, \cdots, x_n=rx_0$ be such that

$$(3.4)\quad \bar S_{x_{i-1}}\subset \bar S_{x_i} \quad\text{and}\quad \max|\mu|\,(\bar S_{x_i}-S_{x_{i-1}})<\delta,\quad (i=1,2,\cdots,n).$$

For the given ε, choose a $\lambda_i\in\Lambda$ for each x_i ($i=0, 1, \cdots, n-1$) so that

$$0\leqq\lambda^*(\bar S_{x_i})-\lambda_i(\bar S_{x_i})\leqq\varepsilon,$$

and for $i\geqq 1$

$$\lambda_i(\bar S_{x_i})-\lambda_{i-1}(\bar S_{x_i})\geqq 0.$$

This is possible from the definition of λ^*.

Now define a function λ_ε as follows: Let $\lambda_\varepsilon(E)=\lambda_{n-1}(E)$ for $E\subset \bar S_{rx_0}-S_{x_{n-1}}$ ($E\in\mathcal{M}_0$). If $\lambda_{n-1}(\bar S_{x_{n-1}})>\lambda_{n-2}(\bar S_{x_{n-1}})$, let x'_{n-2} (if it exists) be such that

$$x'_{n-2}\in S_{x_{n-1}}-\bar S_{x_{n-2}},\qquad \lambda_{n-1}(\bar S_{x'_{n-2}})=\lambda_{n-2}(\bar S_{x'_{n-2}})$$

and if x''_{n-2} also satisfies these conditions then $\bar S_{x'_{n-2}}\supset\bar S_{x''_{n-2}}$. If such an x'_{n-2} does not exist, let $x'_{n-2}=x_{n-2}$. If $\lambda_{n-1}(\bar S_{x_{n-1}})=\lambda_{n-2}(\bar S_{x_{n-1}})$, let $x'_{n-2}=x_{n-1}$. Define

$$\begin{aligned} \lambda_\varepsilon(E) &= \lambda_{n-1}(E) \quad \text{for } E\subset S_{x_{n-1}}-S_{x'_{n-2}}, \\ &= \lambda_{n-2}(E) \quad \text{for } E\subset S_{x'_{n-2}}-S_{x_{n-2}}; \end{aligned}\qquad (E\in\mathcal{M}_0)$$

and

$$\begin{aligned} \lambda_\varepsilon(\bar S_{x'_{n-2}}) &= \lambda_{n-2}(\bar S_{x'_{n-2}}) \quad \text{when } x'_{n-2}\in\bar S_{x_{n-1}}-\bar S_{x_{n-2}}, \\ &= \lambda_{n-1}(\bar S_{x'_{n-2}}) \quad \text{when } x'_{n-2}=x_{n-2}. \end{aligned}$$

If $\lambda_\varepsilon(\bar S_{x_{n-2}})>\lambda_{n-3}(\bar S_{x_{n-2}})$, let x'_{n-3} (if it exists) $\in S_{x_{n-2}}-\bar S_{x_{n-3}}$ be such that

$$\begin{aligned} \lambda_{n-3}(\bar S_{x'_{n-3}}) &= \lambda_{n-2}(\bar S_{x'_{n-3}}) \quad \text{when } x'_{n-2}\in\bar S_{x_{n-1}}-\bar S_{x_{n-2}}, \\ &= \lambda_{n-1}(\bar S_{x'_{n-3}}) \quad \text{when } x'_{n-2}=x_{n-2}; \end{aligned}$$

and if x''_{n-3} be any point with this property then $\bar{S}_{x'_{n-3}} \supset \bar{S}_{x''_{n-3}}$. If such an x'_{n-3} does not exist, let $x'_{n-3}=x_{n-3}$. If $\lambda_\varepsilon(\bar{S}_{x_{n-2}})=\lambda_{n-3}(S_{x_{n-2}})$, let $x'_{n-3}=x_{n-2}$. Define

$$\begin{aligned}\lambda_\varepsilon(E) &= \lambda_{n-1}(E) \quad \text{for } E \subset S_{x_{n-2}} - S_{x'_{n-3}} \quad \text{when } x'_{n-2} = x_{n-2},\\ &= \lambda_{n-2}(E) \quad \text{for } E \subset S_{x_{n-2}} - S_{x'_{n-3}} \quad \text{when } x'_{n-2} \in S_{x_{n-1}} - \bar{S}_{x_{n-2}},\\ &= \lambda_{n-3}(E) \quad \text{for } E \subset S_{x'_{n-3}} - S_{x_{n-3}}\end{aligned}$$

$$(E \in \mathscr{M}_0).$$

Continuing in this way a function λ_ε can be defined on all sets E $(\in \mathscr{M}_0)$ such that E is a subset of one of the sets

$$(3.5) \qquad S_{x'_i} - S_{x_i}, \quad S_{x_{i+1}} - S_{x'_i}, \quad \bar{S}_{rx_0} - S_{x_{n-1}} \quad (i = 0, 1, \cdots, n-2).$$

Also, define $\lambda_\varepsilon(\bar{S}_{x_0})=\alpha_0$. Now extend the definition of λ_ε on $\mathscr{M}_0$ by countable-additivity; i.e. if E $(\in \mathscr{M}_0) \subset \bar{S}_{rx_0} - S_{x_0}$, define

$$(3.6) \qquad \begin{aligned}\lambda_\varepsilon(E) = \sum_{i=0}^{n-2} \lambda_\varepsilon(E \cap (S_{x'_i} - S_{x_i})) + \sum_{i=0}^{n-2} \lambda_\varepsilon(E \cap (S_{x_{i+1}} - S_{x'_i}))\\ + \lambda_\varepsilon(E \cap (\bar{S}_{rx_0} - S_{x_{n-1}})).\end{aligned}$$

Now it can be easily seen that λ_ε is a solution in Λ having the property:

$$(3.7) \qquad 0 \leqq \lambda^*(\bar{S}_{x_i}) - \lambda_\varepsilon(\bar{S}_{x_i}) \leqq \varepsilon \qquad (i = 0, 1, \cdots, n).$$

Furthermore, if E $(\in \mathscr{M}_0)$ is a subset of one of the sets in (3.5), it follows from (3.2), (3.3), (3.4) and (3.7) that

$$(3.8) \qquad 0 \leqq \lambda^*(E) - \lambda_\varepsilon(E) < 3\varepsilon.$$

Letting $\varepsilon = 1/m$ $(m=1, 2, \cdots)$, we obtain a sequence $\lambda_{1/m}$ of solutions such that

$$(3.9) \qquad \lim_{m \to \infty} \lambda_{1/m}(E) = \lambda^*(E)$$

where E $(\in \mathscr{M}_0)$ is a subset of one of the sets in (3.5). For E $(\in \mathscr{M}_0) \subset \bar{S}_{rx_0} - S_{x_0}$, we have

$$(3.10) \qquad \begin{aligned}\lambda^*(E) \geqq \lim_{m \to \infty} \lambda_{1/m}(E) = \sum_{i=0}^{n-2} \lambda^*(E \cap (S_{x'_i} - S_{x_i}))\\ + \sum_{i=0}^{n-2} \lambda^*(E \cap (S_{x_{i+1}} - S_{x'_i})) + \lambda^*(E \cap (\bar{S}_{rx_0} - S_{x_{n-1}})),\end{aligned}$$

by (3.6) and (3.9). Also,

$$\begin{aligned}\lambda^*(E) &= \sup_{\lambda\in\Lambda} \lambda(E)\\ &= \sup_{\lambda\in\Lambda}\left\{\sum_{i=0}^{n-2}\lambda(E\cap(S_{x_i'}-S_{x_i}))+\sum_{i=0}^{n-2}\lambda(E\cap(S_{x_{i+1}}-S_{x_i'}))\right.\\ &\qquad\qquad\left.+\,\lambda(E\cap(\bar{S}_{rx_0}-S_{x_{n-1}}))\right\} \qquad (3.11)\\ &\leqq \sum_{i=0}^{n-2}\lambda^*(E\cap(S_{x_i'}-S_{x_i}))+\sum_{i=0}^{n-2}\lambda^*(E\cap(S_{x_{i+1}}-S_{x_i'}))\\ &\qquad +\,\lambda^*(E\cap(\bar{S}_{rx_0}-S_{x_{n-1}})).\end{aligned}$$

(3.10) and (3.11) yield

$$\lambda^*(E) = \lim_{m\to\infty} \lambda_{1/m}(E) \qquad (E\in\mathcal{M}_0). \tag{3.12}$$

Now, since $\lambda_{1/m}\in\Lambda$,

$$\lambda_{1/m}(E) = \int_E f(x, \lambda_{1/m}(\bar{S}_x))\,d\mu, \qquad E\subset \bar{S}_{rx_0}-S_{x_0} \quad (E\in\mathcal{M}_0).$$

Taking limits as $m\to\infty$, it follows from (3.12) and conditions (ii) and (iii), using Lebesgue's dominated convergence theorem, that

$$\lambda^*(E) = \int_E f(x, \lambda^*(\bar{S}_x))\,d\mu \quad \text{for } E\subset \bar{S}_{rx_0}-S_{x_0} \quad (E\in\mathcal{M}_0).$$

λ^* is thus a maximum solution of (*).

Similarly, if we define $\lambda_*(E)=\inf_{\lambda\in\Lambda}\{\lambda(E)\}$ $(E\in\mathcal{M}_0)$, it can be proved that λ_* is a minimum solution of (*).

This completes the proof.

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