EXISTENCE OF SOLUTIONS OF ABSTRACT MEASURE DIFFERENTIAL EQUATIONS

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Abstract. In an earlier paper the author has introduced an abstract measure differential equation as a generalization of ordinary differential equations and measure differential equations, and proved a theorem for the existence and uniqueness of solutions of this equation. In the present paper the problem of the existence of solutions is further investigated.

1. Introduction. An abstract measure differential equation is defined in [3] as follows. Let \( X \) be a linear space over the field \( \mathcal{F} \) where \( \mathcal{F} \) is the set \( R \) of real numbers or the set \( C \) of complex numbers. For each \( x \in X \), define

\[
S_x = \{ax : -\infty < a < 1\}, \quad S'_x = \{ax : -\infty < a \leq 1\} \quad \text{if } \mathcal{F} = R;
\]

\[
S_x = \{ax : 0 < |a| < 1\}, \quad S'_x = \{ax : 0 \leq |a| \leq 1\} \quad \text{if } \mathcal{F} = C.
\]

Let \( S \subseteq X \) where \( S \) is a proper subset of \( X \), it is of the form \( S_\xi \) for some \( \xi \in X \). Let \( \mathcal{M} \) be a \( \sigma \)-algebra in \( S \) containing the sets \( S_x \) for all \( x \in S \). We shall denote by \( \text{ca}(S, \mathcal{M}) \) the space of all countably additive scalar functions (i.e. real measures or complex measures) on \( \mathcal{M} \). Let \( \Omega \subseteq \mathcal{F} \) be defined by

\[
\Omega = \{x : |x| < a\}.
\]

Abstract measure differential equations are equations of the form

\[
d\lambda/d\mu = f(x, \lambda(S_x))
\]

where \( \mu \) is a positive \( \sigma \)-finite measure or a complex measure on \( \mathcal{M} \), \( d\lambda/d\mu \) is Radon-Nikodym derivative of a measure \( \lambda \in \text{ca}(S, \mathcal{M}) \) with respect to \( \mu \), and \( f \) is a function defined on \( S \times \Omega \) such that \( f(x, \lambda(S_x)) \) is \( \mu \)-integrable on \( S \) for each \( \lambda \in \text{ca}(S, \mathcal{M}) \).

The solution of \((*)\) is defined as follows. Let \( x_0 \in \Omega \), \( x_0 \subseteq S \), \( S_{x_0} \subseteq X_0 \subseteq \mathcal{M} \) and let \( \mathcal{M}_0 \) be the smallest \( \sigma \)-algebra in \( X_0 \) containing \( S_{x_0} - S_{x_0} \) and the sets \( S_x \) for \( x \in X_0 - S_{x_0} \). A measure \( \lambda \in \text{ca}(X_0, \mathcal{M}_0) \) is called a solution of \((*)\).

Received by the editors September 21, 1971 and, in revised form, November 24, 1971.

AMS 1970 subject classifications. Primary 34G05; Secondary 46G99.

Key words and phrases. Abstract measure differential equation, real measures, complex measures, Radon-Nikodym derivative, total variation measure, pseudometric.
on $X_0$ with initial data $[S_x_0, x_0]$ (to be denoted by $\lambda(X_0; S_x_0, a_0)$) if $\lambda(S_x_0) = a_0$, $\lambda(E) \in \Omega$ for $E \in \mathcal{M}_0$, $\lambda \ll \mu$ on $X_0 - S_x_0$ and $\lambda$ satisfies ($*$) a.e. $[\mu]$ on $X_0 - S_x_0$.

An existence and uniqueness theorem is proved in [3] where one of the assumptions for $f$ is to satisfy a Lipschitz condition in $\alpha$. In the present paper the existence of a solution (without claiming its uniqueness) is established by changing some of the assumptions, particularly the Lipschitz condition hypothesis for $f$ is replaced by the continuity of $f$ in $\alpha$ for fixed $x$. Taking $ca(X, \mathcal{M})$ to be the space of real measures, maximum and minimum solutions are defined and their existence is demonstrated under the hypotheses of the existence theorem. It may be noted that Theorem 1 of this paper includes and extends [1, Theorem 1.1, p. 43] and Theorem 2 includes and extends [1, Theorem 1.2, p. 45].

2. Existence of solutions. For any $E_1, E_2 \in \mathcal{M}$, we define

\begin{equation}
\rho(E_1, E_2) = |\mu|(E_1 - E_2) + |\mu|(E_2 - E_1)
\end{equation}

where $|\mu|$ denotes the total variation measure of $\mu$. It is clear that for any $E_1, E_2 \in \mathcal{M}$,

- $\rho(E_1, E_2) \geq 0$,
- $\rho(E_1, E_2) = 0$ if $E_1 = E_2$,
- $\rho(E_1, E_2) = \rho(E_2, E_1)$.

Also, for any $E_1, E_2, E_3 \in \mathcal{M}$, we have

\[\rho(E_1, E_2) \leq \rho(E_1, E_3) + \rho(E_3, E_2),\]

which follows from

- $(E_1 - E_2) \subseteq (E_1 - E_3) \cup (E_3 - E_2)$,
- $(E_2 - E_1) \subseteq (E_2 - E_3) \cup (E_3 - E_2)$.

The function $\rho$ thus defines a pseudometric for $\mathcal{M}$.

We shall now prove the following existence theorem.

**Theorem 1.** Let $\alpha_0 \in \Omega$, $x_0 \in S$ and let for each $\xi \in S - S_x_0$ the smallest $\sigma$-algebra containing $S_{\xi_0} - S_{\xi_0}$ and $S_{\xi}$, $x \in S_x - S_x$ be compact in the topology generated by the pseudometric $\rho$ defined by (2.1). Then there exists a solution $\lambda_0 = \lambda_0[S_x; S_{\xi_0}, \alpha_0]$ of ($*$) for some $x' \in S - S_x$ if the following conditions are satisfied:

(i) $\mu(S_{\xi_0}) \neq 0$ and $|\mu|(S_x - S_x) = 0$ for each $x \in S - S_{\xi_0}$;
(ii) there exists a $\mu$-integrable function $w$ on $S$ such that $|f(x, \alpha)| \leq w(x)$ uniformly in $\alpha \in \Omega$;
(iii) $f$ is continuous in $\alpha$ for each fixed $x$. 


Proof. Choose a real number $r > 1$ such that

\begin{equation}
\int_{S_{r \xi} - S_{\xi_0}} w(x) \, d|\mu| < a - |\alpha_0|.
\end{equation}

It is possible to choose such a real number $r$ (see the proof of Theorem 1 in [3]).

Let $\mathcal{M}_0$ be the smallest $\sigma$-algebra containing $S_{\xi_0} - S_{\xi_0}$ and all the sets of the form $S_x$ for $x \in S_{\xi_0} - S_{\xi_0}$. In what follows, it will always be assumed that $E \in \mathcal{M}_0$. Define measures $\lambda_j$ by

\begin{equation}
\lambda_j(E) = \begin{cases} 
\alpha_0 & \text{for } E = S_{\xi_0}, \\
0 & \text{for } E \subset S_{\xi_0+\xi_j} - S_{\xi_0}, \\
\int_E f(x, \lambda_j(S_{x-\xi_j})) \, d\mu & \text{for } E \subset S_{r \xi_0} - S_{\xi_0+\xi_j},
\end{cases}
\end{equation}

\begin{equation}
\xi_j = [(r - 1)/j]x_0.
\end{equation}

Then $\lambda_j$ is defined by the first two expressions in (2.3). For any fixed $j \geq 2$, the first and second expressions in (2.3) define $\lambda_j$ for $E \subset S_{\xi_0+\xi_j}$; and since $(x, 0)$ and $(x, \alpha_0) \in S_{\xi_0} \times \Omega$, the last expression defines $\lambda_j$ for $E \subset S_{\xi_0+\xi_j} - S_{\xi_0}$, $\lambda_j$ is then defined for $E \subset S_{\xi_0+\xi_j}$. Also, for $E \subset S_{\xi_0+2\xi_j}$, we have

\begin{equation}
|\lambda_j(E)| \leq |\alpha_0| + \int_{S_{r \xi_0} - S_{\xi_0}} w(x) \, d|\mu| < a,
\end{equation}

by condition (ii) and (2.2); and, therefore,

\begin{equation}
\lambda_j(E) \in \Omega.
\end{equation}

Assume that $\lambda_j$ is defined for $E \subset S_{\xi_0+k\xi_j}$ when $2 \leq k < j$. Then the last expression in (2.3) defines $\lambda_j$ for $S_{\xi_0+(k+1)\xi_j} - S_{\xi_0+k\xi_j}$, and thus $\lambda_j$ is defined for all $E \subset S_{\xi_0+(k+1)\xi_j}$. Also, for such $E$'s, $\lambda_j(E)$ satisfies (2.5) and hence (2.6) because of condition (ii) and (2.2). Therefore, by induction, (2.3) defines $\lambda_j$ on $\mathcal{M}_0$.

Let $E_1, E_2 \in \mathcal{M}_0$ be such that $\rho(E_1, E_2) < |\mu|(S_{\xi_0})$. Then $S_{\xi_0}$ is a subset either of both $E_1$ and $E_2$ or of neither of $E_1$ and $E_2$. For, otherwise,

\[ \rho(E_1, E_2) = |\mu|(E_1 - E_2) + |\mu|(E_2 - E_1) \geq |\mu|(S_{\xi_0}). \]

It can be verified that either

\[ \lambda_j(E_1) - \lambda_j(E_2) = 0. \]
or else
\[ \lambda_j(E_1) - \lambda_j(E_2) = \int_{K_1 - K_2} f(x, \lambda_j(S_{x-\xi})) \, d\mu - \int_{E_2 - K_1} f(x, \lambda_j(S_{x-\xi})) \, d\mu, \]
and therefore, by condition (ii),
\[
|\lambda_j(E_1) - \lambda_j(E_2)| \leq \int_{K_1 - K_2} w(x) \, d|\mu| + \int_{E_2 - K_1} w(x) \, d|\mu|
\]
\[= \int_{(E_1 - E_2) \cup (E_2 - E_1)} w(x) \, d|\mu|. \tag{2.7} \]
By Dunford and Schwartz [2, Theorem 20, p. 114], if
\[ v(E) = \int_{E} w(x) \, d|\mu| \]
then
\[ \lim_{|\mu|(E) \to 0} v(E) = 0. \]
This implies that the function \( v \) defined by (2.8) is continuous (and hence uniformly continuous) on the compact pseudometric space \( \mathcal{M}_0 \). From the uniform continuity of \( v \) on \( \mathcal{M}_0 \), it follows from (2.7) that given any \( \varepsilon > 0 \), there exists a \( \delta(\varepsilon) > 0 \) such that \( |\lambda_j(E_1) - \lambda_j(E_2)| < \varepsilon \) whenever \( |\mu|(E_1 - E_2) + |\mu|(E_2 - E_1) < \delta \); i.e. \( \{\lambda_j\} \) is an equicontinuous set. Moreover, \( \{\lambda_j\} \) is uniformly bounded since, by (2.5), we have
\[ \sup_{E \in \mathcal{M}_0} |\lambda_j(E)| \leq \alpha_0 + \int_{S_{\xi_0} - S_{\xi_0}} w(x) \, d|\mu| < a. \tag{2.9} \]
Therefore, it follows by the Arzela-Ascoli theorem [2, p. 266] that \( \{\lambda_j\} \) is conditionally compact in the space \( C(\mathcal{M}_0) \) of all bounded continuous scalar functions on \( \mathcal{M}_0 \). Hence there exists a subsequence \( \{\lambda_{j_k}\} \) of \( \{\lambda_j\} \) and a function \( \lambda_0 \in C(\mathcal{M}_0) \) such that \( \lambda_{j_k} \to \lambda_0 \) uniformly on \( \mathcal{M}_0 \) as \( k \to \infty \). And since
\[ \rho(S_{x-\xi_{j_k}}, S) = |\mu| \, (S_{x} - S_{x-\xi_k}) \to |\mu| \, (S_{x} - S_{2}) = 0, \]
by condition (i) it follows that \( \lambda_{j_k}(S_{x-\xi_{j_k}}) \to \lambda_0(S_{2}) \). Also, by (2.9), \( \lambda_0(E) \in \Omega \) for \( E \in \mathcal{M}_0 \). Hence the continuity of \( f \) in \( x \) for fixed \( x \) (condition (iii)) implies
\[ \lim_{k \to x} f(x, \lambda_{j_k}(S_{x-\xi_k})) = f(x, \lambda_0(S_{2})). \tag{2.10} \]
From (2.10) and condition (ii), it follows by Lebesgue's dominated convergence theorem that
\[ \lim_{k \to x} \int_{E} f(x, \lambda_{j_k}(S_{x-\xi_k})) \, d\mu = \int_{E} f(x, \lambda_0(S_{2})) \, d\mu. \]
Now replacing \( j \) by \( j_k \) in (2.3) and letting \( k \to \infty \), we obtain

\[
\lambda_0(E) = \alpha_0 \quad \text{for } E = S_{x_0},
\]

(2.11)

\[
\lambda_0(E) = \int_E f(x, \lambda_0(S_x)) \, d\mu \quad \text{for } E \subset S_{x_0} - S_{x_0}.
\]

Thus \( \lambda_0 \) satisfies (\( * \)) a.e. \([\mu]\) on \( S_{x_0} - S_{x_0} \). It also follows from (2.11) that \( \lambda_0 \) is countably additive on \( \mathcal{M}_0 \). Hence \( \lambda_0 \) is a solution of (\( * \)) on \( S_{x_0} \) with initial condition \( \lambda_0(S_{x_0}) = \alpha_0 \). This completes the proof.

### 3. Maximum and minimum solutions

In this section we shall be concerned with real measures only and so \( \mathcal{CA}(S, \mathcal{M}) \) will now denote the space of real measures on \( \mathcal{M} \). Let \( \alpha_0, x_0, X_0 \) and \( \mathcal{M}_0 \) be the same as in §1. Let \( \Lambda \) be the set of all solutions of (\( * \)) on \( X_0 \) with initial data \([S_{x_0}, \alpha_0]\).

**Definition.** A maximum solution of (\( * \)) on \( X_0 \) with initial data \([S_{x_0}, \alpha_0]\) is a solution \( A \in \Lambda \) with the property

\[
A(E) \leq A_0(E) \quad (E \in \mathcal{M}_0)
\]

for each \( A \in \Lambda \). Similarly, \( \lambda_m \in \Lambda \) will be called a minimum solution of (\( * \)) on \( X_0 \) with initial data \([S_{x_0}, \alpha_0]\) if

\[
A(E) \geq A_m(E) \quad (E \in \mathcal{M}_0)
\]

for each \( A \in \Lambda \).

We shall now establish the existence of \( A_0 \) and \( A_m \) under the assumptions of Theorem 1.

**Theorem 2.** Let the hypotheses of Theorem 1 be satisfied. Then there exist a maximum solution \( A_0 \) and a minimum solution \( A_m \) of (\( * \)) on \( X_0 = S_{x_0} \) (\( S_{x_0} \) being defined by (2.2)) with initial data \([S_{x_0}, \alpha_0]\).

**Proof.** Define \( A_0(E) = \sup_{A \in \Lambda} \{A(E)\} \quad (E \in \mathcal{M}_0) \). We shall prove that \( A_0 \) is a maximal solution \( \lambda_0 \).

Clearly \( \lambda_0(S_{x_0}) = \alpha_0 \) and \( \lambda_0 \ll \mu \) on \( X_0 - S_{x_0} \). By condition (i) of Theorem 1, each \( \lambda \in \Lambda \) satisfies the inequality

\[
|\lambda(E)| \leq |\alpha_0| + \int_{S_{x_0} - S_{x_0}} w(x) \, d|\mu| \quad (E \in \mathcal{M}_0).
\]

(3.1)

And, since

\[
|\lambda_0(E)| = \sup_{\lambda \in \Lambda} |\lambda(E)| \leq \sup_{\lambda \in \Lambda} |\lambda(E)| \quad (E \in \mathcal{M}_0).
\]

it follows from (3.1) and (2.2) that \( \lambda_0(E) \in \Omega \) for each \( E \in \mathcal{M}_0 \).

We shall now show that \( \lambda_0 \) satisfies (\( * \)) a.e. \([\mu]\) on \( X_0 - S_{x_0} \). It can be shown, in the same way as the equicontinuity of \( \{\lambda_j\} \) was demonstrated in the proof of Theorem 1, that \( \Lambda \) is an equicontinuous set. Thus, given any
\( \varepsilon > 0 \), there exists a \( \delta = \delta(\varepsilon) > 0 \) such that \( E_1, E_2 \in \mathcal{M}_0 \) and \( \rho(E_1, E_2) < \delta \) imply

\[
|\lambda(E_1) - \lambda(E_2)| < \varepsilon \quad \text{for each } \lambda \in \Lambda,
\]
and hence

\[
|\lambda^*(E_1) - \lambda^*(E_2)| = \left| \sup_{\lambda \in \Lambda} \lambda(E_1) - \sup_{\lambda \in \Lambda} \lambda(E_2) \right| \\
\leq \left| \sup_{\lambda \in \Lambda} (\lambda(E_1) - \lambda(E_2)) \right| \\
\leq \sup_{\lambda \in \Lambda} |\lambda(E_1) - \lambda(E_2)| \leq \varepsilon.
\]

Let \( x_1, x_2, \ldots, x_n = r x_0 \) be such that

\[
S_{x_{i-1}} \subset S_{x_i} \quad \text{and} \quad \max |\mu| (S_{x_i} - S_{x_{i-1}}) < \delta, \quad (i = 1, 2, \cdots, n).
\]

For the given \( \varepsilon \), choose a \( \lambda_i \in \Lambda \) for each \( x_i \) (\( i = 0, 1, \cdots, n-1 \)) so that

\[
0 \leq \lambda_i(S_{x_i}) - \lambda_i(S_{x_i}) \leq \varepsilon,
\]
and for \( i \geq 1 \)

\[
\lambda_i(S_{x_i}) - \lambda_{i-1}(S_{x_i}) \geq 0.
\]

This is possible from the definition of \( \lambda^* \).

Now define a function \( \lambda^*_n \) as follows: Let \( \lambda^*_n(E) = \lambda_{n-1}(E) \) for \( E \subset S_{x_{n-1}} \subset S_{x_n} \) (\( E \in \mathcal{M}_0 \)). If \( \lambda_{n-1}(S_{x_{n-1}}) > \lambda_{n-2}(S_{x_{n-1}}) \), let \( x'_n \) (if it exists) be such that

\[
x'_n = S_{x_{n-1}} - S_{x_n} \quad \quad \lambda_{n-1}(S_{x'_n}) = \lambda_{n-2}(S_{x'_n})
\]
and if \( x''_n \) also satisfies these conditions then \( S_{x''_n} \supset S_{x'_n} \). If such an \( x''_n \) does not exist, let \( x''_n = x_{n-2} \). If \( \lambda_{n-1}(S_{x_{n-1}}) = \lambda_{n-2}(S_{x_{n-1}}) \), let \( x'_n = x_{n-2} \). Define

\[
\lambda_n(E) = \begin{cases} 
\lambda_{n-1}(E) & \text{for } E \subset S_{x_{n-1}} - S_{x_{n-2}}, \\
\lambda_{n-2}(E) & \text{for } E \subset S_{x_{n-2}} - S_{x_{n-1}}, 
\end{cases} \quad (E \in \mathcal{M}_0)
\]
and

\[
\lambda_n(S_{x_{n-1}}) = \begin{cases} 
\lambda_{n-2}(S_{x_{n-2}}) & \text{when } x'_n \in S_{x_{n-1}} - S_{x_{n-2}}, \\
\lambda_{n-1}(S_{x_{n-2}}) & \text{when } x'_n = x_{n-2}.
\end{cases}
\]
If \( \lambda_n(S_{x_{n-1}}) > \lambda_{n-2}(S_{x_{n-2}}) \), let \( x''_n \) (if it exists) be such that

\[
\lambda_{n-2}(S_{x''_n}) = \lambda_{n-2}(S_{x'_n}) \quad \text{when } x''_n \in S_{x_{n-1}} - S_{x_{n-2}},
\]
\[= \lambda_{n-1}(S_{x''_n}) \quad \text{when } x''_n = x_{n-2}.
\]
and if $x_{n-3}$ be any point with this property then \( S_{x_{n-3}} \supset S_{x_{n-3}} \). If such an $x_{n-3}$ does not exist, let $x_{n-3} = x_{n-3}$. If $\lambda(E)(S_{x_{n-3}}) = \lambda_{n-3}(S_{x_{n-3}})$, let $x_{n-3} = x_{n-2}$.

Define

\[
\lambda(E) = \begin{cases} 
\lambda_{n-1}(E) & \text{for } E \subseteq S_{x_{n-2}} - S_{x_{n-3}} \text{ when } x_{n-2} = x_{n-2}, \\
\lambda_{n-2}(E) & \text{for } E \subseteq S_{x_{n-2}} - S_{x_{n-3}} \text{ when } x_{n-2} \in S_{x_{n-1}} - S_{x_{n-2}}, \\
\lambda_{n-3}(E) & \text{for } E \subseteq S_{x_{n-2}} - S_{x_{n-3}} 
\end{cases} 
\]

\((E \in \mathcal{M}_0)\).

Continuing in this way a function $\lambda_e$ can be defined on all sets $E (\in \mathcal{M}_0)$ such that $E$ is a subset of one of the sets

\[(3.5) \quad S_{x_i} - S_{x_{i+1}} - S_{x_{i+2}} - \cdots - S_{x_{n-1}} - S_{x_n} \quad (i = 0, 1, \cdots, n - 2).\]

Also, define $\lambda_0(S_{x_0}) = x_0$. Now extend the definition of $\lambda_e$ on $\mathcal{M}_0$ by countable-additivity; i.e. if $E (\in \mathcal{M}_0) \subseteq S_{x_0} - S_{x_0}$, define

\[(3.6) \quad \lambda(E) = \sum_{i=0}^{n-2} \lambda_i(E \cap (S_{x_i} - S_{x_{i+1}})) + \sum_{i=0}^{n-2} \lambda_i(E \cap (S_{x_{i+1}} - S_{x_{i+2}})) + \lambda(E \cap (S_{x_0} - S_{x_{n-1}})).\]

Now it can be easily seen that $\lambda_e$ is a solution in $\Lambda$ having the property:

\[(3.7) \quad 0 \leq \lambda_*(S_{x_i}) - \lambda_*(S_{x_{i+1}}) \leq \varepsilon \quad (i = 0, 1, \cdots, n).\]

Furthermore, if $E (\in \mathcal{M}_0)$ is a subset of one of the sets in (3.5), it follows from (3.2), (3.3), (3.4) and (3.7) that

\[(3.8) \quad 0 \leq \lambda_*(E) - \lambda_*(E) < 3\varepsilon.\]

Letting $\varepsilon = 1/m \quad (m = 1, 2, \cdots)$, we obtain a sequence $\lambda_{1/m}$ of solutions such that

\[(3.9) \quad \lim_{m \to \infty} \lambda_{1/m}(E) = \lambda_*(E)\]

where $E (\in \mathcal{M}_0)$ is a subset of one of the sets in (3.5). For $E (\in \mathcal{M}_0) \subseteq S_{x_0} - S_{x_0}$, we have

\[(3.10) \quad \lambda_*(E) = \lim_{m \to \infty} \lambda_{1/m}(E) = \sum_{i=0}^{n-2} \lambda_*(E \cap (S_{x_i} - S_{x_{i+1}}))
+ \sum_{i=0}^{n-2} \lambda_*(E \cap (S_{x_{i+1}} - S_{x_{i+2}})) + \lambda_*(E \cap (S_{x_0} - S_{x_{n-1}})).\]
by (3.6) and (3.9). Also,

\[ \lambda^*(E) = \sup_{\Lambda} \lambda(E) \]

\[ = \sup_{\Lambda} \left\{ \sum_{i=0}^{n-2} \lambda(E \cap (S_{x_i} - S_{x_{i+1}})) + \sum_{i=0}^{n-2} \lambda(E \cap (S_{x_{i+1}} - S_{x_i})) + \lambda(E \cap (S_{r_{x_0}} - S_{x_{n-1}})) \right\} \]

(3.11)

\[ \sup_{\Lambda} \left\{ \sum_{i=0}^{n-2} \lambda^*(E \cap (S_{x_i} - S_{x_{i+1}})) + \sum_{i=0}^{n-2} \lambda^*(E \cap (S_{x_{i+1}} - S_{x_i})) + \lambda^*(E \cap (S_{r_{x_0}} - S_{x_{n-1}})) \right\} \]

(3.10) and (3.11) yield

(3.12) \[ \lambda^*(E) = \lim_{m \to \infty} \lambda_{1/m}(E) \quad (E \in M_0). \]

Now, since \( \lambda_{1/m} \in \Lambda \),

\[ \lambda_{1/m}(E) = \int_{E'} f(x, \lambda_{1/m}(S_x)) \, d\mu, \quad E \subseteq S_{r_{x_0}} - S_{x_m} \quad (E \in M_0). \]

Taking limits as \( m \to \infty \), it follows from (3.12) and conditions (ii) and (iii), using Lebesgue’s dominated convergence theorem, that

\[ \lambda^*(E) = \int_{E'} f(x, \lambda^*(S_x)) \, d\mu \quad \text{for} \quad E \subseteq S_{r_{x_0}} - S_{x_0} \quad (E \in M_0). \]

\( \lambda^* \) is thus a maximum solution of \((*)\).

Similarly, if we define \( \lambda_\star(E) = \inf_{\Lambda} \{ \lambda(E) \} \) \( (E \in M_0) \), it can be proved that \( \lambda_\star \) is a minimum solution of \((*)\).

This completes the proof.

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