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THE FINITENESS OF I WHEN $R[X]/I$ IS R-FLAT. II

WILLIAM HEINZER AND JACK OHM[1]

ABSTRACT. This paper supplements work of Ohm-Rush. A question which was raised by them is whether $R[X]/I$ is a flat R-module implies I is locally finitely generated at primes of $R[X]$. Here R is a commutative ring with identity, X is an indeterminate, and I is an ideal of $R[X]$. It is shown that this is indeed the case, and it then follows easily that I is even locally principal at primes of $R[X]$.

Ohm-Rush have also observed that a ring R with the property "$R[X]/I$ is R-flat implies I is finitely generated" is necessarily an $A(0)$ ring, i.e. a ring such that finitely generated flat modules are projective; and they have asked whether conversely any $A(0)$ ring has this property. An example is given to show that this conjecture needs some tightening. Finally, a theorem of Ohm-Rush is applied to prove that any R with only finitely many minimal primes has the property that $R[X]/I$ is R-flat implies I is finitely generated.

Notation. All rings will be commutative with identity. R will always denote a ring, X an indeterminate, and I an ideal in $R[X]$. If $f \in R[X]$, the content of f, $c(f)$, is the ideal of R generated by the coefficients of f; and if I is an ideal of $R[X]$, $c(I)$ denotes the ideal of R generated by the coefficients of the elements of I. If R' is an R-algebra with defining homomorphism $\phi: R \to R'$ and A' is an ideal of R', then we use $A' \cap R$ to denote the ideal $\phi^{-1}(A')$. R' is called a simple R-algebra if ϕ extends to a surjective homomorphism $\phi_X: R[X] \to R'$; if $\xi = \phi_X(X)$, we write $R' = R[\xi]$.

1. I is locally finitely generated.

The theorem of this section has been proved by Ohm-Rush [**OR**, Theorem 2.18] under the assumption that I contains a regular element whose degree is minimal among the nonzero elements of I.

THEOREM 1.1. *Let I be an ideal in the polynomial ring $R[X]$. If $R[X]/I$ is a flat R-module, then for any prime ideal P of $R[X]$, $IR[X]_P$ is principal.*

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PROOF. It suffices to show that $IR[X]_P$ is finitely generated, for as pointed out in [**OR**, Proposition 1.6] principalness is then an easy consequence of Nakayama's lemma. Note also that one need only consider the case that $I \subset P$.

Our proof requires a number of preliminary reductions.

(a) Reduction to the case that R is quasi-local and P contracts to the maximal ideal of R. If $p = P \cap R$, by localizing with respect to the multiplicative system $R \setminus p$ we may assume that R is quasi-local and that P contracts to the maximal ideal p of R. We use here a fact that recurs throughout the paper, namely that if R' is any R-algebra, then $0 \to I \to R[X] \to R[X]/I \to 0$ is exact and $R[X]/I$ is R-flat imply $0 \to IR'[X] \to R'[X] \to R'[X]/IR'[X] \to 0$ is exact and $R'[X]/IR'[X]$ is R'-flat [**B**, p. 30, Proposition 4 and p. 34, Corollary 2].

(b) Passage from a quasi-local ring R, p to a Henselian quasi-local ring with infinite residue field. Let R, p be a quasi-local ring and let R', p' be a quasi-local ring such that R' is a faithfully flat R-algebra and $pR' = p'$. Then $R'[X] = R' \otimes_R R[X]$ is a faithfully flat $R[X]$-module [**B**, p. 48, Proposition 5]. Hence if P is a prime ideal of $R[X]$, then there exists a prime ideal P' in $R'[X]$ lying over P; and if, moreover, $P \cap R = p$, then $P' \cap R' = p'$ since $pR' = p'$. Also, $R'[X]_{P'}$ is a faithfully flat $R[X]_P$-module. A consequence of this faithful flatness is that any ideal in $R[X]_P$ extends and contracts to itself in $R'[X]_{P'}$ [**B**, p. 51, Proposition 9], and hence an ideal in $R[X]_P$ is finitely generated if and only if its extension to $R'[X]_{P'}$ is finitely generated. Thus, if I is an ideal in $R[X]$ and P is a prime of $R[X]$ such that $P \cap R = p$ and $I \subset P$, then there is a prime ideal P' of $R'[X]$ lying over P such that $P' \cap R' = p'$ and $IR[X]_P$ is finitely generated if and only if $IR'[X]_{P'}$ is finitely generated.

There are two rings to which we want to apply the above remarks. First let $R' = R(Y)$, where $R(Y)$ denotes the ring $R[Y]_S$, Y an indeterminate and $S = \{f \in R[Y] \mid c(f) = R\}$. If R, p is quasi-local, then $R(Y)$ is quasi-local with maximal ideal $pR(Y)$ and has infinite residue field [**N**, p. 18]. Moreover, $R(Y)$ is a flat and hence faithfully flat R-module. Thus, by replacing the ring R, p by $R(Y), pR(Y)$, we may assume that R, p has infinite residue field.

The next reduction involves passing to the Henselization. If R, p is quasi-local, then the Henselization R^* of R is quasi-local with maximal ideal $pR^* = p^*$, $R/p = R^*/p^*$, and R^* is a faithfully flat R-module [**N**, p. 180, (43.3) and p. 182, (43.8)]. The above remarks show that we may replace R, p by its Henselization and thus may assume that R, p is a Henselian quasi-local ring with infinite residue field.

(c) Reduction to the case that I contains a polynomial $g(X)$ with $g(0) = 1$. We note first that R, p is quasi-local and $R[X]/I$ is R-flat imply either

$I=(0)$ or $I\not\subset pR[X]$ [**OR**, Corollary 1.3] or [**B**, p. 66, Example 23-d]. Thus, excluding the trivial case that $I=(0)$, there exists $g(X)\in I$ with $g(X)\notin pR[X]$. Since R/p is infinite, there exists $a\in R$ such that $g(a)\not\equiv 0 \pmod p$; and hence $g(a)$ is a unit of R. Let ϕ be the R-automorphism of $R[X]$ defined by $\phi(X)=X+a$. Since $\phi(g)(0)=g(a)$, we may, after replacing I by $\phi(I)$, assume that $g(0)$ is a unit of R. After dividing g by $g(0)$, we may further assume $g(0)=1$.

The above reductions show that it suffices to prove the following proposition.

PROPOSITION 1.2. *Let R, p be a Henselian quasi-local ring; let $S=R[X]\backslash P$, where P is a prime ideal of $R[X]$ such that $P\cap R=p$; and let I be an ideal of $R[X]$ such that $I\subset P$ and I contains a polynomial $g(X)$ with $g(0)=1$. Then $R[X]/I$ is a flat R-module implies I_S is a finitely generated ideal of $R[X]_S$.*

First we need a couple of easy lemmas. Recall that an R-algebra R' is said to be of *finite type* if R' is a localization of a finite R-algebra [**N**, p. 127].[2]

LEMMA 1.3. *Let R, p be a Henselian quasi-local ring and let R', p' be a quasi-local R-algebra of finite type such that $p'\cap R=p$. Then R' is a finite R-module.*

PROOF. By definition R' is a localization of a finite R-algebra T. It follows that $R'=T_Q$, where $Q=p'\cap T$. Since R is Henselian, $T=\prod_{i=1}^n T_i$, where the T_i are quasi-local [**N**, p. 185, (43.15)]. Note that $p'\cap R=p$ implies $Q\cap R=p$, and since T is integral over R, this implies that Q is maximal. But the maximal ideals of $\prod_{i=1}^n T_i$ are all of the form $(T_1,\cdots,Q_i,\cdots,T_n)$, where Q_i is the maximal ideal of T_i, and $\prod_{i=1}^n T_i$ localized at any such prime is merely a homomorphic image of $\prod_{i=1}^n T_i$. Thus, T is a finite R-module implies T_Q is a finite R-module.

LEMMA 1.4. *Let R, p be a Henselian quasi-local ring, let $g(X)\in R[X]$ be a polynomial such that $g(0)=1$, let P be a prime ideal of $R[X]$ with $P\cap R=p$ and $g\in P$, and let $\phi: R[X]\to R[X]/(g(X))$ denote the canonical homomorphism. Then $(R[X]/(g(X)))_{\phi(P)}$ is a finite R-module.*

PROOF. If $\xi=\phi(X)$, then $R[X]/(g(X))=R[\xi]$. Since $g(0)=1$, ξ is a unit in $R[\xi]$ and $1/\xi$ is integral over R. Thus $R[\xi]_{\phi(P)}$ is a localization of $R[1/\xi]$ and is therefore a quasi-local R-algebra of finite type with $\phi(P)R[\xi]_{\phi(P)}\cap R=p$. By 1.3, $R[\xi]_{\phi(P)}$ is a finite R-module. q.e.d.

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[2] This differs from Bourbaki's terminology. Probably "essentially finite" would be a better name for this kind of R-algebra.

PROOF OF 1.2. Consider the exact sequence of $R[X]$-modules

$$0 \to I/(g) \to R[X]/(g) \to R[X]/I \to 0.$$

Localizing at the multiplicative system S, we get the exact sequence

$$(1.5) \qquad 0 \to (I/(g))_S \to (R[X]/(g))_S \to (R[X]/I)_S \to 0.$$

By Lemma 1.4, $(R[X]/(g))_S$ is a finite R-module and hence so also is $(R[X]/I)_S$. Moreover, $R[X]/I$ is R-flat implies $(R[X]/I)_S$ is R-flat. Therefore $(R[X]/I)_S$ is a finite flat R-module; and since R is quasi-local, this implies $(R[X]/I)_S$ is R-free. Thus the sequence (1.5) splits and $(I/(g))_S$ is also R-finite and a fortiori $R[X]_S$-finite. Since $I_S/(gR[X]_S)$ is canonically isomorphic as an $R[X]_S$-module to $(I/(g))_S$, we conclude that I_S is a finite $R[X]_S$-module. q.e.d.

Let us call an ideal A of a ring R *locally trivial* if for every prime p of R, either $A_p=0$ or $A_p=R_p$.

COROLLARY 1.6. *Let I be an ideal of $R[X]$. Then $R[X]/I$ is R-flat if and only if $c(I)$ is locally trivial and I is locally principal at primes of $R[X]$.*

PROOF. Apply Theorem 1.1 and [**OR**, Theorem 1.5 and Proposition 1.6].

COROLLARY 1.7. *If I is an ideal in $R[X]$ such that $R[X]/I$ is R-flat, then I is a flat $R[X]$-module.*

PROOF. It follows from Corollary 1.6 that I is locally free at each prime of $R[X]$.

COROLLARY 1.8. *Let I and J be ideals of $R[X]$. If $R[X]/I$ and $R[X]/J$ are R-flat, then $R[X]/IJ$ is R-flat.*

PROOF. Note that IJ is locally principal and $c(IJ)$ is locally trivial. Hence Corollary 1.6 applies.

COROLLARY 1.9. *Let R be a ring, let $\bar{R}$ denote the integral closure of R in its total quotient ring, and let I be an ideal of $R[X]$. Then $R[X]/I$ is R-flat if (and only if) $\bar{R}[X]/I\bar{R}[X]$ is $\bar{R}$-flat.*

PROOF. The proof is the same as in [**OR**, Theorem 2.18], except that Theorem 1.1 is used in place of their Corollary 2.16.

2. **Flatness and $A(0)$ rings.** We shall call a ring R an $A(0)$ ring (in keeping with the terminology of [**CP**]) provided finitely generated flat R-modules are projective. R is an $A(0)$ ring if and only if every locally trivial ideal A of R is finitely generated [**OR**, Lemma 4.6]; and an immediate consequence of this and the definition is that R is an $A(0)$ ring if and only if for every ideal A of R, R/A is R-flat implies A is finitely generated.

Consider the following assertion:

(*) $R[X]/I$ is a flat R-module implies I is finitely generated.

It is proved in [**OR**, Theorem 2.19] that if R is a domain then (*) is always valid; moreover, the existence of rings which are not $A(0)$ rings (e.g. absolutely flat rings which are not noetherian) shows that (*) is not true in general without some assumption on I or R. The question is raised in [**OR**] as to what rings R have the property that (*) is valid for all ideals I of $R[X]$, and Ohm and Rush suggest that (*) might be true whenever R is an $A(0)$ ring. This possibility is supported by their observation that R is $A(0)$ if and only if for every ideal I of $R[X]$, $R[X]/I$ is a finite flat R-module implies I is finitely generated (which shows a fortiori that (*) implies R is an $A(0)$ ring). We shall give now an example of a quasi-local ring (and hence an $A(0)$ ring) for which (*) does not hold. The idea behind the example is to reduce to a ring which is not $A(0)$ by localizing at an element s. Thus, perhaps the rings for which (*) is valid are those R with the property that simple flat R-algebras are $A(0)$. The following lemma shows that this condition is at least necessary.

LEMMA 2.1. *If R satisfies* (*), *then any simple flat R-algebra is an* $A(0)$ *ring.*

PROOF. Suppose there exists a simple flat R-algebra $R[\xi]$ which is not $A(0)$. Then there exists an ideal A of $R[\xi]$ such that $R[\xi]/A$ is $R[\xi]$-flat but A is not finitely generated. By [**B**, p. 35, Corollary 3], $R[\xi]/A$ is also R-flat. If I denotes the kernel of the composition of the canonical homomorphisms $R[X]\to R[\xi]\to R[\xi]/A$, then the image of I in $R[\xi]$ is A; and hence I cannot be finitely generated. Thus, R does not satisfy (*).

EXAMPLE 2.2 (of a quasi-local ring R and an ideal I of $R[X]$ such that $R[X]/I$ is R-flat but I is not finitely generated).

Claim. There exists an integral domain D with the following properties.

(i) D is 2-dimensional quasi-local;

(ii) the maximal ideal of D is the radical of a principal ideal;

(iii) the set $\{p_\alpha\}$ of all height one primes of D is infinite and $\bigcap_\alpha p_\alpha \neq (0)$.

Before verifying the claim, let us show how the existence of such a D leads to the required example. Let $N=\bigcap_\alpha p_\alpha$, and let $R=D/N$. Then R is quasi-local, reduced, 1-dimensional and the maximal ideal of R is of the form $\sqrt{(s)}$ for some $s\in R$. Moreover, R has an infinite number of minimal primes. It follows that $R[1/s]$ is 0-dimensional, reduced, and has an infinite number of minimal primes, where $R[1/s]$ denotes the quotient ring of R with respect to the multiplicative system consisting of powers of s. Therefore $R[1/s]$ is absolutely flat and nonnoetherian, so $R[1/s]$ is not an

$A(0)$ ring. Hence by Lemma 2.1, there exists an ideal I of $R[X]$ such that $R[X]/I$ is R-flat but I is not finitely generated.

Note that in the above argument the fact that $\sqrt{(s)}$ is the maximal ideal of R is used only to insure that $R[1/s]$ has infinitely many primes. Thus, for our application it would be sufficient to have infinitely many minimal primes of R which do not contain s.

We shall now prove the above claim. Let k be an algebraically closed field of characteristic zero and let y and z be indeterminates. Let $K=k(y, z)$ and define a rank two valuation ring V of K over k by defining $V(y)=(0, 1)$, $V(z)=(1, 0)$ and then taking infimums, i.e. the value of any polynomial in $k[y, z]$ is the infimum of the values of the monomials occurring in that polynomial. Here the value group for V is the direct sum of two copies of the additive group of integers ordered lexicographically. Thus, $V(y)<V(z)$. Note that V has maximal ideal yV, $V=k+yV$, and the z-adic valuation ring of $k(y, z)$, viz., $k[y, z]_{(z)}$, is the rank one valuation ring of K containing V. Let L be an algebraic closure of K and let V^* denote the integral closure of V in L. Since V^* is a Prüfer domain (see for example, [**G**, p. 257, (18.3)] or [**K**, p. 71, Theorem 101]) each extension of the valuation ring $k[y, z]_{(z)}$ to L is of the form $V^*_{P_\alpha}$ for some height one prime P_α of V^*. It is easily seen that there are infinitely many valuation rings of L extending $k[y, z]_{(z)}$ (for example, if θ is a root of the polynomial X^n-1+z, then in $K(\theta)$ there are n valuation rings extending $k[y, z]_{(z)}$). Thus, the set $\{P_\alpha\}$ of height one primes of V^* is infinite. Let M denote the Jacobson radical of V^* and let $D=k+M$. We have $V\subset D\subset V^*$, so V^* is integral over D. Hence D is 2-dimensional quasi-local with maximal ideal M, $M=\sqrt{(yD)}$, and each height one prime of D is of the form $p_\alpha=P_\alpha\cap D$. We note that $1/y\in D_{p_\alpha}$ and $yV^*\subset M\subset D$, so $V^*\subset D_{p_\alpha}$ and $D_{p_\alpha}=V^*_{P_\alpha}$. Therefore the set $\{p_\alpha\}$ of height one primes of D is infinite. Finally, $z\in\bigcap_\alpha p_\alpha$, so $\bigcap_\alpha p_\alpha\neq(0)$, and D has all the properties of the claim. q.e.d.

The following proposition shows that if R is a ring with nilradical N and if R/N satisfies condition $(*)$ introduced above then R does also.

PROPOSITION 2.3. *Let N be the nilradical of the ring R, let I be an ideal of $R[X]$, and assume $R[X]/I$ is R-flat. Then I is a finitely generated ideal if (and only if) the image of I under the canonical homomorphism $R[X]\to(R/N)[X]$ is a finitely generated ideal.*

PROOF. The hypotheses imply there exists a finitely generated ideal $A\subset I$ such that $I=A+(NR[X]\cap I)$. Since $R[X]/I$ is R-flat, $NR[X]\cap I=NI$ [**B**, p. 33, Corollary]. If P is any prime ideal of $R[X]$, then $I_P=A_P+NI_P$; and since I_P is finitely generated by Theorem 1.1, it follows from Nakayama's lemma that $I_P=A_P$. Therefore, $I=A$. q.e.d.

We now prove a theorem which gives a large class of rings that do satisfy (*). The proof will make use of Theorem 1.1 and the result of Ohm-Rush that integral domains satisfy (*).

THEOREM 2.4. *Let R be a ring with only finitely many minimal prime ideals, and let I be an ideal of $R[X]$. Then $R[X]/I$ is R-flat implies I is a finitely generated ideal.*

PROOF. If $p_1, \cdots, p_n$ are the minimal primes of R, then the canonical homomorphism $R \to \prod_{i=1}^n (R/p_i) = R'$ defines an R-algebra structure on R'. Since $R' = Re_1 + \cdots + Re_n$, where $e_i = (0, \cdots, 1_i, \cdots, 0)$, R' is a finite R-module. If I_i denotes the image of I under the canonical homomorphism $R[X] \to (R/p_i)[X]$, then $IR'[X] = \prod_{i=1}^n I_i$. Moreover, since R/p_i is a domain [**OR**, Theorem 2.19] asserts that I_i is a finitely generated ideal. It follows that $IR'[X]$ is a finitely generated ideal of $R'[X]$. Therefore, there exists a finitely generated ideal A of $R[X]$ such that $A \subset I$ and $AR'[X] = IR'[X]$.

The remainder of the proof is essentially the same as the proof of [**OR**, Theorem 2.19]. For a given prime P of $R[X]$, we show that $AR[X]_P = IR[X]_P$. If $P \cap R = p$, we may localize at $R \backslash p$ and thus assume that R is quasi-local with maximal ideal p. Let p' be a prime of R' lying over p. Then $(R/p)[X] \subset (R'/p')[X]$ and $A(R'/p')[X] = I(R'/p')[X]$. Since $(R/p)[X]$ is a principal ideal domain, it follows that $A(R/p)[X] = I(R/p)[X]$. Hence $I = A + (I \cap p[X])$; and since $R[X]/I$ is R-flat, $I \cap p[X] = pI$ [**B**, p. 33, Corollary]. Thus $I = A + pI$, so $IR[X]_P = AR[X]_P + pIR[X]_P$. Since $IR[X]_P$ is finitely generated (Theorem 1.1), Nakayama's lemma implies that $AR[X]_P = IR[X]_P$. We conclude that $A = I$. q.e.d.

The final part of the proof of the above theorem actually yields the following result, which is perhaps of interest in itself.

PROPOSITION 2.5. *Suppose R' is an R-algebra such that every prime ideal of R has a prime ideal of R' lying over it, and let $A \subset I$ be ideals of $R[X]$ such that $R[X]/I$ is R-flat. Then $AR'[X] = IR'[X]$ implies $A = I$.*

ADDED FEBRUARY 7, 1972. That $R[X]/I$ is R-flat implies I is locally finitely generated at primes of $R[X]$ is known and is due to M. Raynaud and L. Gruson, "Critères de platitude et de projectivité", Invent. Math. **13** (1971), 1–89. In fact, their 3.4.2 contains the n-variable case of this theorem! Similarly, their 3.4.6 includes the n-variable analog of our 2.4 (for a reduced R). We are indebted to W. Vasconcelos for directing our attention to this important paper. While the methods of Raynaud-Gruson are very impressive, we feel that our proofs retain some interest because of their accessibility.

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Department of Mathematics, Purdue University, Lafayette, Indiana 47907 (Current address of William Heinzer)

Current address (Jack Ohm): Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803