THE FINITENESS OF I WHEN R[X]/I IS R-FLAT. II

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ABSTRACT. This paper supplements work of Ohm-Rush. A question which was raised by them is whether R[X]/I is a flat R-module implies I is locally finitely generated at primes of R[X]. Here R is a commutative ring with identity, X is an indeterminate, and I is an ideal of R[X]. It is shown that this is indeed the case, and it then follows easily that I is even locally principal at primes of R[X].

Ohm-Rush have also observed that a ring R with the property "R[X]/I is R-flat implies I is finitely generated" is necessarily an A(0) ring, i.e., a ring such that finitely generated flat modules are projective; and they have asked whether conversely any A(0) ring has this property. An example is given to show that this conjecture needs some tightening. Finally, a theorem of Ohm-Rush is applied to prove that any R with only finitely many minimal primes has the property that R[X]/I is R-flat implies I is finitely generated.

Notation. All rings will be commutative with identity. R will always denote a ring, X an indeterminate, and I an ideal in R[X]. If \( f \in R[X] \), the content of \( f \), c(\( f \)), is the ideal of R generated by the coefficients of \( f \); and if I is an ideal of R[X], c(I) denotes the ideal of R generated by the coefficients of the elements of I. If R' is an R-algebra with defining homomorphism \( \phi : R \to R' \) and A' is an ideal of R', then we use A' \cap R to denote the ideal \( \phi^{-1}(A') \). R' is called a simple R-algebra if \( \phi \) extends to a surjective homomorphism \( \phi_X : R[X] \to R' \); if \( \xi = \phi_X(X) \), we write \( R' = R[\xi] \).

1. I is locally finitely generated. The theorem of this section has been proved by Ohm-Rush [OR, Theorem 2.18] under the assumption that I contains a regular element whose degree is minimal among the nonzero elements of I.

THEOREM 1.1. Let I be an ideal in the polynomial ring R[X]. If R[X]/I is a flat R-module, then for any prime ideal P of R[X], IR[X]_I, is principal.

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PROOF. It suffices to show that $IR[X]_P$ is finitely generated, for as pointed out in \cite[Proposition 1.6]{OR} principalness is then an easy consequence of Nakayama's lemma. Note also that one need only consider the case that $I \subseteq P$.

Our proof requires a number of preliminary reductions.

(a) Reduction to the case that $R$ is quasi-local and $P$ contracts to the maximal ideal of $R$. If $p = P \cap R$, by localizing with respect to the multiplicative system $R \setminus p$ we may assume that $R$ is quasi-local and that $P$ contracts to the maximal ideal $p$ of $R$. We use here a fact that recurs throughout the paper, namely that if $R'$ is any $R$-algebra, then $0 \to I \to R[X] \to R[X]/I \to 0$ is exact and $R[X]/I$ is $R$-flat imply $0 \to I R'[X] \to R'[X] \to R'[X]/I R'[X] \to 0$ is exact and $R'[X]/I R'[X]$ is $R'$-flat \cite[p. 30, Proposition 4 and p. 34, Corollary 2]{B}.

(b) Passage from a quasi-local ring $R, p$ to a Henselian quasi-local ring with infinite residue field. Let $R, p$ be a quasi-local ring and let $R', p'$ be a quasi-local ring such that $R'$ is a faithfully flat $R$-algebra and $p R' = p'$. Then $R'[X] = R' \otimes_R R[X]$ is a faithfully flat $R[X]$-module \cite[p. 48, Proposition 5]{B}. Hence if $P$ is a prime ideal of $R[X]$, then there exists a prime ideal $P'$ in $R'[X]$ lying over $P$; and if, moreover, $P \cap R = p$, then $P' \cap R' = p'$ since $p R' = p'$. Also, $R'[X]_{P'}$ is a faithfully flat $R[X]_{P'}$-module. A consequence of this faithful flatness is that any ideal in $R[X]_P$ extends and contracts to itself in $R'[X]_{P'}$ \cite[p. 51, Proposition 9]{B}, and hence an ideal in $R[X]_P$ is finitely generated if and only if its extension to $R'[X]_{P'}$ is finitely generated. Thus, if $I$ is an ideal in $R[X]$ and $P$ is a prime of $R[X]$ such that $P \cap R = p$ and $I \subseteq P$, then there is a prime ideal $P'$ of $R'[X]$ lying over $P$ such that $P' \cap R' = p'$ and $I R[X]_P$ is finitely generated if and only if $I R'[X]_{P'}$ is finitely generated.

There are two rings to which we want to apply the above remarks. First let $R' = R(Y)$, where $R(Y)$ denotes the ring $R[Y][f]$, $Y$ an indeterminate and $S = \{ f \in R[Y] \mid c(f) = R \}$. If $R, p$ is quasi-local, then $R(Y)$ is quasi-local with maximal ideal $p R(Y)$ and has infinite residue field \cite[p. 18]{N}. Moreover, $R(Y)$ is a flat and hence faithfully flat $R$-module. Thus, by replacing the ring $R, p$ by $R(Y), p R(Y)$, we may assume that $R, p$ has infinite residue field.

The next reduction involves passing to the Henselization. If $R, p$ is quasi-local, then the Henselization $R^*$ of $R$ is quasi-local with maximal ideal $p R^* = p^*$, $R/p = R^*/p^*$, and $R^*$ is a faithfully flat $R$-module \cite[p. 180, (43.3) and p. 182, (43.8)]{N}. The above remarks show that we may replace $R, p$ by its Henselization and thus may assume that $R, p$ is a Henselian quasi-local ring with infinite residue field.

(c) Reduction to the case that $I$ contains a polynomial $g(X)$ with $g(0) = 1$. We note first that $R, p$ is quasi-local and $R[X]/I$ is $R$-flat imply either
$I = (0)$ or $I \notin p \mathcal{R}[X]$ [OR, Corollary 1.3] or [B, p. 66, Example 23-d]. Thus, excluding the trivial case that $I = (0)$, there exists $g(X) \in I$ with $g(X) \notin p \mathcal{R}[X]$. Since $R/p$ is infinite, there exists $a \in R$ such that $g(a) \not\equiv 0 \pmod{p}$; and hence $g(a)$ is a unit of $R$. Let $\phi$ be the $R$-automorphism of $\mathcal{R}[X]$ defined by $\phi(X) = X + a$. Since $\phi(g)(0) = g(a)$, we may, after replacing $I$ by $\phi(I)$, assume that $g(0)$ is a unit of $R$. After dividing $g$ by $g(0)$, we may further assume $g(0) = 1$.

The above reductions show that it suffices to prove the following proposition.

**Proposition 1.2.** Let $R$, $p$ be a Henselian quasi-local ring; let $S = \mathcal{R}[X]/P$, where $P$ is a prime ideal of $\mathcal{R}[X]$ such that $P \cap R = p$; and let $I$ be an ideal of $\mathcal{R}[X]$ such that $I \subseteq p$ and $I$ contains a polynomial $g(X)$ with $g(0) = 1$. Then $\mathcal{R}[X]/I$ is a flat $R$-module implies $I_S$ is a finitely generated ideal of $\mathcal{R}[X]_S$.

First we need a couple of easy lemmas. Recall that an $R$-algebra $R'$ is said to be of finite type if $R'$ is a localization of a finite $R$-algebra [N, p. 127].

**Lemma 1.3.** Let $R$, $p$ be a Henselian quasi-local ring and let $R'$, $p'$ be a quasi-local $R$-algebra of finite type such that $p' \cap R = p$. Then $R'$ is a finite $R$-module.

**Proof.** By definition $R'$ is a localization of a finite $R$-algebra $T$. It follows that $R' = T_Q$, where $Q = p' \cap T$. Since $R$ is Henselian, $T = \prod_{i=1}^{n} T_i$, where the $T_i$ are quasi-local [N, p. 185, (43.15)]. Note that $p' \cap R = p$ implies $Q \cap R = p$, and since $T$ is integral over $R$, this implies that $Q$ is maximal. But the maximal ideals of $\prod_{i=1}^{n} T_i$ are all of the form $(T_1, \ldots, Q_i, \ldots, T_n)$, where $Q_i$ is the maximal ideal of $T_i$, and $\prod_{i=1}^{n} T_i$ localized at any such prime is merely a homomorphic image of $\prod_{i=1}^{n} T_i$. Thus, $T$ is a finite $R$-module implies $T_Q$ is a finite $R$-module.

**Lemma 1.4.** Let $R$, $p$ be a Henselian quasi-local ring, let $g(X) \in \mathcal{R}[X]$ be a polynomial such that $g(0) = 1$, let $P$ be a prime ideal of $\mathcal{R}[X]$ with $P \cap R = p$ and $g \in P$, and let $\phi: \mathcal{R}[X] \to \mathcal{R}[X]/(g(X))$ denote the canonical homomorphism. Then $\mathcal{R}[X]/(g(X))_{\phi(\mathcal{R}[X])}$ is a finite $R$-module.

**Proof.** If $\xi = \phi(X)$, then $\mathcal{R}[X]/(g(X)) = \mathcal{R}[\xi]$. Since $g(0) = 1$, $\xi$ is a unit in $\mathcal{R}[\xi]$ and $1/\xi$ is integral over $R$. Thus $\mathcal{R}[\xi]_{\phi(\mathcal{R}[X])}$ is a localization of $R[1/\xi]$ and is therefore a quasi-local $R$-algebra of finite type with $\phi(P)R[\xi]_{\phi(\mathcal{R}[X])} \cap R = p$. By 1.3, $R[\xi]_{\phi(\mathcal{R}[X])}$ is a finite $R$-module. q.e.d.

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2 This differs from Bourbaki's terminology. Probably "essentially finite" would be a better name for this kind of $R$-algebra.
Proof of 1.2. Consider the exact sequence of \( R[X] \)-modules
\[
0 \to I/(g) \to R[X]/(g) \to R[X]/I \to 0.
\]
Localizing at the multiplicative system \( S \), we get the exact sequence
\[
(1.5) \quad 0 \to (I/(g))_S \to (R[X]/(g))_S \to (R[X]/I)_S \to 0.
\]
By Lemma 1.4, \((R[X]/(g))_S\) is a finite \( R \)-module and hence so also is \((R[X]/I)_S\). Moreover, \( R[X]/I \) is \( R \)-flat implies \((R[X]/I)_S\) is \( R \)-flat. Therefore \((R[X]/I)_S\) is a finite flat \( R \)-module; and since \( R \) is quasi-local, this implies \((R[X]/I)_S\) is \( R \)-free. Thus the sequence (1.5) splits and \((I/(g))_S\) is also \( R \)-finite and a fortiori \((R[X]/I)_S\)-finite. Since \( I_S/(gR[X])_S \) is canonically isomorphic as an \( R[X]_S \)-module to \((I/(g))_S\), we conclude that \( I_S \) is a finite \( R[X]_S \)-module. q.e.d.

Let us call an ideal \( A \) of a ring \( R \) locally trivial if for every prime \( p \) of \( R \), either \( A_p = 0 \) or \( A_p = R_p \).

Corollary 1.6. Let \( I \) be an ideal of \( R[X] \). Then \( R[X]/I \) is \( R \)-flat if and only if \( c(I) \) is locally trivial and \( I \) is locally principal at primes of \( R[X] \).

Proof. Apply Theorem 1.1 and [OR, Theorem 1.5 and Proposition 1.6].

Corollary 1.7. If \( I \) is an ideal in \( R[X] \) such that \( R[X]/I \) is \( R \)-flat, then \( I \) is a flat \( R[X] \)-module.

Proof. It follows from Corollary 1.6 that \( I \) is locally free at each prime of \( R[X] \).

Corollary 1.8. Let \( I \) and \( J \) be ideals of \( R[X] \). If \( R[X]/I \) and \( R[X]/J \) are \( R \)-flat, then \( R[X]/IJ \) is \( R \)-flat.

Proof. Note that \( IJ \) is locally principal and \( c(IJ) \) is locally trivial. Hence Corollary 1.6 applies.

Corollary 1.9. Let \( R \) be a ring, let \( \bar{R} \) denote the integral closure of \( R \) in its total quotient ring, and let \( I \) be an ideal of \( R[X] \). Then \( R[X]/I \) is \( R \)-flat if (and only if) \( \bar{R}[X]/I\bar{R}[X] \) is \( \bar{R} \)-flat.

Proof. The proof is the same as in [OR, Theorem 2.18], except that Theorem 1.1 is used in place of their Corollary 2.16.

2. Flatness and \( A(0) \) rings. We shall call a ring \( R \) an \( A(0) \) ring (in keeping with the terminology of [CP]) provided finitely generated flat \( R \)-modules are projective. \( R \) is an \( A(0) \) ring if and only if every locally trivial ideal \( A \) of \( R \) is finitely generated [OR, Lemma 4.6]; and an immediate consequence of this and the definition is that \( R \) is an \( A(0) \) ring if and only if for every ideal \( A \) of \( R \), \( R/A \) is \( R \)-flat implies \( A \) is finitely generated.
Consider the following assertion:

\((\ast)\) \quad R[X]/I is a flat \(R\)-module implies \(I\) is finitely generated.

It is proved in [OR, Theorem 2.19] that if \(R\) is a domain then \((\ast)\) is always valid; moreover, the existence of rings which are not \(A(0)\) rings (e.g. absolutely flat rings which are not noetherian) shows that \((\ast)\) is not true in general without some assumption on \(I\) or \(R\). The question is raised in [OR] as to what rings \(R\) have the property that \((\ast)\) is valid for all ideals \(I\) of \(R[X]\), and Ohm and Rush suggest that \((\ast)\) might be true whenever \(R\) is an \(A(0)\) ring. This possibility is supported by their observation that \(R\) is \(A(0)\) if and only if for every ideal \(I\) of \(R[X]\), \(R[X]/I\) is a flat \(R\)-module implies \(I\) is finitely generated (which shows a fortiori that \((\ast)\) implies \(R\) is an \(A(0)\) ring). We shall give now an example of a quasi-local ring (and hence an \(A(0)\) ring) for which \((\ast)\) does not hold. The idea behind the example is to reduce to a ring which is not \(A(0)\) by localizing at an element \(s\). Thus, perhaps the rings for which \((\ast)\) is valid are those \(R\) with the property that simple flat \(R\)-algebras are \(A(0)\). The following lemma shows that this condition is at least necessary.

**Lemma 2.1.** If \(R\) satisfies \((\ast)\), then any simple flat \(R\)-algebra is an \(A(0)\) ring.

**Proof.** Suppose there exists a simple flat \(R\)-algebra \(R[\xi]\) which is not \(A(0)\). Then there exists an ideal \(A\) of \(R[\xi]\) such that \(R[\xi]/A\) is \(R[\xi]\)-flat but \(A\) is not finitely generated. By [B, p. 35, Corollary 3], \(R[\xi]/A\) is also \(R\)-flat. If \(I\) denotes the kernel of the composition of the canonical homomorphisms \(R[X] \to R[\xi] \to R[\xi]/A\), then the image of \(I\) in \(R[\xi]\) is \(A\); and hence \(I\) cannot be finitely generated. Thus, \(R\) does not satisfy \((\ast)\).

**Example 2.2** (of a quasi-local ring \(R\) and an ideal \(I\) of \(R[X]\) such that \(R[X]/I\) is \(R\)-flat but \(I\) is not finitely generated).

**Claim.** There exists an integral domain \(D\) with the following properties.

(i) \(D\) is 2-dimensional quasi-local;
(ii) the maximal ideal of \(D\) is the radical of a principal ideal;
(iii) the set \(\{p_s\}\) of all height one primes of \(D\) is infinite and \(\bigcap_s p_s \neq (0)\).

Before verifying the claim, let us show how the existence of such a \(D\) leads to the required example. Let \(N = \bigcap_s p_s\), and let \(R = D/N\). Then \(R\) is quasi-local, reduced, 1-dimensional and the maximal ideal of \(R\) is of the form \(\sqrt{(s)}\) for some \(s \in R\). Moreover, \(R\) has an infinite number of minimal primes. It follows that \(R[1/s]\) is 0-dimensional, reduced, and has an infinite number of minimal primes, where \(R[1/s]\) denotes the quotient ring of \(R\) with respect to the multiplicative system consisting of powers of \(s\). Therefore \(R[1/s]\) is absolutely flat and nonnoetherian, so \(R[1/s]\) is not an
A(0) ring. Hence by Lemma 2.1, there exists an ideal I of \( R[X] \) such that \( R[X]/I \) is R-flat but I is not finitely generated.

Note that in the above argument the fact that \((s)\) is the maximal ideal of \( R \) is used only to insure that \( R[1/s] \) has infinitely many primes. Thus, for our application it would be sufficient to have infinitely many minimal primes of \( R \) which do not contain \( s \).

We shall now prove the above claim. Let \( k \) be an algebraically closed field of characteristic zero and let \( y \) and \( z \) be indeterminates. Let \( K = k(y, z) \) and define a rank two valuation ring \( V \) of \( K \) over \( k \) by defining \( V(y) = (0, 1) \), \( V(z) = (1, 0) \) and then taking infimums, i.e. the value of any polynomial in \( k[y, z] \) is the infimum of the values of the monomials occurring in that polynomial. Here the value group for \( V \) is the direct sum of two copies of the additive group of integers ordered lexicographically. Thus, \( V(y) < V(z) \). Note that \( V \) has maximal ideal \( yV \), \( V = k + yV \), and the \( z \)-adic valuation ring of \( k(y, z) \), viz., \( k[y, z]_{(z)} \), is the rank one valuation ring of \( K \) containing \( V \). Let \( L \) be an algebraic closure of \( K \) and let \( V^* \) denote the integral closure of \( V \) in \( L \). Since \( V^* \) is a Prüfer domain (see for example, [G, p. 257, (18.3)] or [K, p. 71, Theorem 101]) each extension of the valuation ring \( k[y, z]_{(z)} \) to \( L \) is of the form \( V^*_P \), for some height one prime \( P \) of \( V^* \). It is easily seen that there are infinitely many valuation rings of \( L \) extending \( k[y, z]_{(z)} \) (for example, if \( \theta \) is a root of the polynomial \( X^n - 1 + z \), then in \( K(\theta) \) there are \( n \) valuation rings extensions \( k[y, z]_{(z)} \)). Thus, the set \( \{ P \} \) of height one primes of \( V^* \) is infinite. Let \( M \) denote the Jacobson radical of \( V^* \) and let \( D = k + M \). We have \( V \subset D \subset V^* \), so \( V^* \) is integral over \( D \). Hence \( D \) is 2-dimensional quasi-local with maximal ideal \( M = \sqrt{\langle yD \rangle} \), and each height one prime of \( D \) is of the form \( P = P \cap D \). We note that \( 1/y \in D_{P} \) and \( yV^* \subset M = D \), so \( V^* \subset D_{P} \) and \( D_{P} = V^*_P \). Therefore the set \( \{ P \} \) of height one primes of \( D \) is infinite. Finally, \( z \in \bigcap{P} \), so \( \bigcap{P} \neq (0) \), and \( D \) has all the properties of the claim. q.e.d.

The following proposition shows that if \( R \) is a ring with nilradical \( N \) and if \( R/N \) satisfies condition (\( \ast \)) introduced above then \( R \) does also.

**Proposition 2.3.** Let \( N \) be the nilradical of the ring \( R \), let \( I \) be an ideal of \( R[X] \), and assume \( R[X]/I \) is \( R \)-flat. Then \( I \) is a finitely generated ideal if (and only if) the image of \( I \) under the canonical homomorphism \( R[X] \rightarrow (R/N)[X] \) is a finitely generated ideal.

**Proof.** The hypotheses imply there exists a finitely generated ideal \( A \subset I \) such that \( I = A + (NR[X] \cap I) \). Since \( R[X]/I \) is \( R \)-flat, \( NR[X] \cap I = NI \) [B, p. 33, Corollary]. If \( P \) is any prime ideal of \( R[X] \), then \( I_P = A_P + MIP \); and since \( I_P \) is finitely generated by Theorem 1.1, it follows from Nakaya-MA's lemma that \( I_P = A_P \). Therefore, \( I = A \). q.e.d.
We now prove a theorem which gives a large class of rings that do satisfy (*). The proof will make use of Theorem 1.1 and the result of Ohm-Rush that integral domains satisfy (*).

**Theorem 2.4.** Let $R$ be a ring with only finitely many minimal prime ideals, and let $I$ be an ideal of $R[X]$. Then $R[X]/I$ is $R$-flat implies $I$ is a finitely generated ideal.

**Proof.** If $p_1, \ldots, p_n$ are the minimal primes of $R$, then the canonical homomorphism $R \rightarrow \prod_{i=1}^n (R/p_i) = R'$ defines an $R$-algebra structure on $R'$. Since $R' = R_{e_1} + \cdots + R_{e_n}$, where $e_i = (0, \ldots, 1_i, \ldots, 0)$, $R'$ is a finite $R$-module. If $I_i$ denotes the image of $I$ under the canonical homomorphism $R[X] \rightarrow (R/p_i)[X]$, then $IR'[X] = \prod_{i=1}^n I_i$. Moreover, since $R/p_i$ is a domain [OR, Theorem 2.19] asserts that $I_i$ is a finitely generated ideal. It follows that $IR'[X]$ is a finitely generated ideal of $R'[X]$. Therefore, there exists a finitely generated ideal $A$ of $R[X]$ such that $A \subseteq I$ and $AR'[X] = IR'[X]$. The remainder of the proof is essentially the same as the proof of [OR, Theorem 2.19]. For a given prime $P$ of $R[X]$, we show that $AR[X]_P = IR[X]_P$. If $P \cap R = p$, we may localize at $R/p$ and thus assume that $R$ is quasi-local with maximal ideal $p$. Let $p'$ be a prime of $R'$ lying over $p$. Then $(R/p)[X] \subseteq (R'/p')[X]$ and $A(R'/p')[X] = IR'(p'/p)[X]$. Since $(R/p)[X]$ is a principal ideal domain, it follows that $A(R/p)[X] = I(R/p)[X]$. Hence $I = A + (I \cap p[X])$; and since $R[X]/I$ is $R$-flat, $I \cap p[X] = pl$ [B, p. 33, Corollary]. Thus $I = A + pl$, so $IR[X]_P = AR[X]_P + pl IR[X]_P$. Since $IR[X]_P$ is finitely generated (Theorem 1.1), Nakayama's lemma implies that $AR[X]_P = IR[X]_P$. We conclude that $A = I$. q.e.d.

The final part of the proof of the above theorem actually yields the following result, which is perhaps of interest in itself.

**Proposition 2.5.** Suppose $R'$ is an $R$-algebra such that every prime ideal of $R$ has a prime ideal of $R'$ lying over it, and let $A \subseteq I$ be ideals of $R[X]$ such that $R[X]/I$ is $R$-flat. Then $AR'[X] = IR'[X]$ implies $A = I$.

**Added February 7, 1972.** That $R[X]/I$ is $R$-flat implies $I$ is locally finitely generated at primes of $R[X]$ is known and is due to M. Raynaud and L. Gruson, "Critères de platitude et de projectivité", Invent. Math. 13 (1971), 1–89. In fact, their 3.4.2 contains the $n$-variable case of this theorem! Similarly, their 3.4.6 includes the $n$-variable analog of our 2.4 (for a reduced $R$). We are indebted to W. Vasconcelos for directing our attention to this important paper. While the methods of Raynaud-Gruson are very impressive, we feel that our proofs retain some interest because of their accessibility.
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