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ON THE LENGTH OF GAPS IN THE ESSENTIAL SPECTRUM OF A GENERALISED DIRAC OPERATOR

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ABSTRACT. The object of the paper is to give an upper bound for the length of the gaps that can occur in the essential spectrum of any selfadjoint operator which is generated by a generalised Dirac system of differential expressions in the Hilbert space $L^2(a, b)$. An estimate is also obtained for the limit point of the spectrum which has least absolute value.

1. Let τ denote the matrix differential expression of order $2n$ given by

$$\tau\varphi(t) = B\varphi'(t) + \Omega(t)\varphi(t), \qquad -\infty \leqq a < t < b \leqq \infty,$$

where B is the real $2n \times 2n$ matrix

$$B = \begin{pmatrix} & I \\ -I & \end{pmatrix}, \qquad I = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix},$$

$\Omega(t)$ is a real symmetric $2n \times 2n$ matrix for each $t \in (a, b)$ and φ is a C^{2n}-valued function on (a, b) having a derivative φ'. Our aim in this paper is to study the selfadjoint operators which are generated by τ in the Hilbert space $L^2_{2n}(a, b)$ of C^{2n}-valued Lebesgue measurable functions φ which satisfy

$$\|\varphi\|^2 = \int_a^b |\varphi(t)|^2\, dt < \infty,$$

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$|\cdot|$ denoting the Euclidean norm on C^{2n}. The inner product on $L^2_{2n}(a, b)$ is written

$$(\varphi, \psi) = \int_a^b \varphi(t) \cdot \psi(t)\, dt$$

where $\varphi(t) \cdot \psi(t)$ is the inner product $\sum_{i=1}^{2n} \varphi_i(t)\bar{\psi}_i(t)$ on C^{2n}.

Let T_0 be the operator defined by $T_0\varphi = \tau\varphi$ on the space of C^{2n}-valued functions φ which have a continuous first derivative in (a, b) and whose support is a compact subset of the open interval (a, b). For a suitable Ω this operator T_0 is a symmetric operator in $L^2_{2n}(a, b)$ as τ is formally selfadjoint, and the closure of T_0 is the minimal operator generated by τ.

The expression τ can be singular at one or both of the end points of (a, b). However our discussion does require the interval to be infinite and so it will always be assumed that at least one of the end points is infinite. In every case the equation $(\tau - \lambda)\varphi = 0$ has only a finite number of $L^2_{2n}(a, b)$ solutions for im $\lambda \neq 0$ which means that T_0 has finite (and of course equal) deficiency indices. This in turn implies that every selfadjoint extension T of T_0 has the same essential spectrum σ_E. Our concern in this paper is with the nature of this essential spectrum, in particular with determining bounds for the length of any gap that can occur in σ_E. We also obtain a bound for the limit point of σ_E which is nearest the origin.

In [4] Weidmann, *inter alia*, obtains results on the essential spectrum of any selfadjoint extension T of T_0 in the case when $n=1$, τ is regular at a and $b = \infty$. As a corollary to our main theorem we obtain a result which is similar to Theorem 6.10 in [4]. Unlike that of Weidmann our result holds for any infinite interval (a, b) and the conditions on Ω are only imposed in a sequence of intervals in (a, b) whose lengths tend to infinity. However, although it is easy to find matrices Ω which satisfy our criterion and not Weidmann's our result is not stronger than his. A more detailed discussion of these results will be postponed until §2. The general system in this paper was studied by Gasymov in [3] in his work on the inverse scattering problem but with $a=0$, $b=\infty$. In order to give a satisfactory solution to the inverse scattering problem Gasymov assumes that the matrix-valued function Ω is B selfadjoint, i.e. for any $t \in (a, b)$, $B\Omega(t)$ is a symmetric matrix. This is equivalent to taking Ω in the form

$$\Omega = \begin{pmatrix} P & QI \\ IQ & -IPI \end{pmatrix} \tag{1.1}$$

where $P(t)$, $Q(t)$ are real symmetric $n \times n$ matrices. He does however show that the problem with a general Ω is unitarily equivalent to one with a B selfadjoint Ω if $|\Omega|$, the norm of Ω in C^{2n}, is assumed to be integrable over $[0, \infty)$. This integrability condition on Ω is too strong for our needs, although we shall make use of the notion of B selfadjointness in our discussion.

2. Before we state and prove our main results it is convenient at this stage to collect some simple facts about B selfajdoint matrices.

Since $B^*=-B$, a matrix Ω_0 is B selfadjoint if and only if $\Omega_0 B=-B\Omega_0$. Hence, since $(\Omega_0+\lambda)Bv=-B(\Omega_0-\lambda)v$ and B is nonsingular ($B^2=-E_{2n}$), it follows that if λ is an eigenvalue of Ω_0 with eigenvector v, then $-\lambda$ is also an eigenvalue of Ω_0 with eigenvector Bv. The eigenvalues of Ω_0 can thus be denoted $\pm\mu_i$, $i=1, 2, \cdots, n$ (repeated according to multiplicity). If e_i is a normalised eigenvector of Ω_0 corresponding to μ_i, Be_i is a normalised eigenvector for $-\mu_i$. Suppose that the set $\{e_1, \cdots, e_n, Be_1, \cdots, Be_n\}$ is orthonormal. The unitary matrix $U=(e_1, \cdots, e_n, Be_n, \cdots, Be_1)$ diagonalises Ω_0,

$$U^*\Omega_0 U = dg(\mu_1, \cdots, \mu_n, -\mu_n, \cdots, -\mu_1)$$

and further $BU=-UB$ so that

$$U^*BU = -B. \tag{2.1}$$

Lastly, if Ω is any real symmetric matrix then $\Omega+B\Omega B$ is B selfadjoint.

Henceforth we shall denote by T any selfajdoint extension of T_0 and we denote by (Λ, Λ') a gap in the essential spectrum σ_E of T. We shall also write ξ_i for the vector $\{\delta_{ji}\}$ in C^{2n}, where δ_{ji} is the Kronecker delta. We assume throughout that $|\Omega|$ is square integrable on a sequence of intervals Δ_m.

THEOREM 1. *Let $\Delta_m=[c_m-a_m, c_m+a_m]$ be intervals in (a, b) with $a_m\to\infty$ as $m\to\infty$. Let Ω_0 be a real B selfadjoint constant matrix with eigenvalues $\pm\mu_j$ $(j=1, 2, \cdots, n)$, $\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_n$, and orthonormal eigenvectors e_j, Be_j, $j=1, 2, \cdots, n$.*

If $\frac{1}{2}(\Lambda'+\Lambda)\in \complement(-\mu_j, \mu_j)$ then

$$\Lambda'-\Lambda \leqq 2\liminf_{m\to\infty}\frac{1}{(2a_m)^{1/2}}\|\{|(\Omega-\Omega_0)e_j|+|(\Omega-\Omega_0)Be_j|\}\|_{\Delta_m} \tag{2.2}$$

so that $\sigma_E\supset \complement(-\mu_j, \mu_j)$ if

$$\liminf_{m\to\infty}\frac{1}{(2a_m)^{1/2}}\|\{|(\Omega-\Omega_0)e_j|+|(\Omega-\Omega_0)Be_j|\}\|_{\Delta_m}=0. \tag{2.3}$$

If $\frac{1}{2}(\Lambda'+\Lambda)\in(-\mu_j, \mu_j)$ then

$$\Lambda'\leqq\mu_j+\liminf_{m\to\infty}\frac{1}{(2a_m)^{1/2}}\|(\Omega-\Omega_0)e_j\|_{\Delta_m}. \tag{2.4}$$

$$\Lambda\geqq-\mu_j-\liminf_{m\to\infty}\frac{1}{(2a_m)^{1/2}}\|(\Omega-\Omega_0)Be_j\|_{\Delta_m}. \tag{2.5}$$

Note that if Ω_0 is diagonal and μ_j is in the sth place along the diagonal then $e_j=\xi_s$, $Be_j=\pm\xi_{2n+1-s}$. Also, the latter results give a bound for the closest point of σ_E to the origin. For if Λ_0 is such a point the gap $(-|\Lambda|_0, |\Lambda_0|)$ in σ_E has midpoint 0 and so

$$
|\Lambda_0| \leqq \min_{1\leqq j\leqq n}\left\{\mu_j + \min\left(\liminf_{m\to\infty}\frac{1}{(2a_m)^{1/2}}\|(\Omega-\Omega_0)e_j\|_{A_m}, \liminf_{m\to\infty}\frac{1}{(2a_m)^{1/2}}\|(\Omega-\Omega_0)Be_j\|_{A_m}\right)\right\}. \tag{2.6}
$$

PROOF. We have remarked above that the constant unitary matrix $U=(e_1,\cdots,e_n, Be_n,\cdots,Be_1)$ satisfies

$$
U^*\Omega_0U = dg(\mu_1,\cdots,\mu_n,-\mu_n,\cdots,-\mu_1) = D
$$

say, and $U^*BU=-B$. Hence T is unitarily equivalent to the operator $\tilde{T}$ which is generated by $\tilde{\tau}$, where

$$
\tilde{\tau}\varphi = U^*\tau U\varphi = -B\varphi' + D\varphi + U^*(\Omega-\Omega_0)U\varphi.
$$

It is therefore sufficient to prove the result for $\tilde{T}$.

The proof depends on the construction of a sequence of C^{2n}-valued functions $\varphi^{(m)}$ which lie in the domain of T_0 and converge weakly to zero. The idea behind such a sequence comes from the work of Eastham on similar problems concerning single differential expressions (see **[1]**, **[2]**).

Let u_m be a real-valued function having a continuous first derivative and satisfying

$$
\begin{aligned}
u_m(t) &= 1 \quad \text{for } |t| \leqq a_m - 1,\\
&= 0 \quad \text{for } |t| \geqq a_m,
\end{aligned}
$$

and $0\leqq u_m(t)\leqq 1$ for all t. Let $\lambda=\frac{1}{2}(\Lambda'+\Lambda)$ and take $|\lambda|>\mu_j$. We define

$$
\begin{aligned}
\varphi^{(m)}(t,\lambda) &= k_m\exp(if_m(t,\lambda))u_m(t-c_m)v \quad \text{for } |t-c_m|\leqq a_m,\\
&= 0 \quad \text{otherwise},
\end{aligned}
$$

where k_m is a positive real constant making $\|\varphi^{(m)}\|=1$, f_m is a real-valued function and v a vector in C^{2n} with $|v|=1$; f_m and v will be chosen later. We have, as $m\to\infty$,

$$
k_m \sim (2a_m)^{-1/2}, \qquad |\varphi^{(m)}(t,\lambda)| \leqq k_m. \tag{2.7}
$$

From these equations it follows that $\varphi^{(m)}\rightharpoonup 0$ as $m\to\infty$.

Now,

$$
\begin{aligned}
(\tilde{\tau}-\lambda)\varphi^{(m)} &= (-if'_mB + D - \lambda)\varphi^{(m)}\\
&\quad - k_m\exp(if_m)u'_m(\cdot-c_m)Bv + U^*(\Omega-\Omega_0)U\varphi^{(m)}.
\end{aligned} \tag{2.8}
$$

We choose f_m and v to satisfy

$$(-if'_m B + D - \lambda)v = 0.$$

It is readily shown that

$$\det(-if'_m B + D - \lambda) = \prod_{j=1}^{n} \{(\lambda^2 - \mu_j^2) - (f'_m)^2\}$$

and so we may choose $(f'_m)^2 = \lambda^2 - \mu_j^2$ and $f_m(t, \lambda) = (\lambda^2 - \mu_j^2)^{1/2}(t - c_m + a_m)$ for $t \in A_m$ and $f_m(t, \lambda) = 0$ otherwise. As $|\lambda| > \mu_j$, f_m is real. For v we may choose $v = \{v_i\}$ where $v_i = 0$ for $i \neq j$, $2n+1-j$ and

$$(\mu_j - \lambda)v_j - if'_m v_{2n+1-j} = 0,$$
$$-(\mu_j + \lambda)v_{2n+1-j} + if'_m v_j = 0.$$

Let $v_j = ((\lambda + \mu_j)/2|\lambda|)^{1/2}$, $v_{2n+1-j} = ((\mu_j - \lambda)/2|\lambda|)^{1/2}$, so that $|v| = 1$ as required. Note that

$$Uv = v_j U\xi_j + v_{2n+1-j} U\xi_{2n+1-j} = v_j e_j + v_{2n+1-j} Be_j$$

so that

$$|U^*(\Omega - \Omega_0)Uv| = |(\Omega - \Omega_0)Uv| \leqq |(\Omega - \Omega_0)e_j| + |(\Omega - \Omega_0)Be_j|.$$

Substituting in (2.8) and using (2.7), we get, since $\varphi^{(m)}$ is continuously differentiable with support in A_m,

$$\begin{aligned} \|(\tilde{T} - \lambda)\varphi^{(m)}\| &= \|(\tilde{\tau} - \lambda)\varphi^{(m)}\| \leqq k_m \|u'_m(\cdot - c_m)\|_{A_m} \\ &\quad + k_m \|\{|(\Omega - \Omega_0)e_j| + |(\Omega - \Omega_0)Be_j|\}\|_{A_m} \\ &= O(k_m) + k_m \|\{|(\Omega - \Omega_0)e_j| + |(\Omega - \Omega_0)Be_j|\}\|_{A_m}. \end{aligned} \tag{2.9}$$

As σ_E has a gap (Λ, Λ') then for any δ, $0 < \delta < \frac{1}{2}(\Lambda' - \Lambda)$, $(\Lambda + \delta, \Lambda' - \delta)$ contains only a finite number of eigenvalues of finite multiplicity of T and hence also of $\tilde{T}$. Let $\lambda_1, \cdots, \lambda_N$ be these eigenvalues of $\tilde{T}$, repeated according to multiplicity, and let $\psi_1, \cdots, \psi_N$ be an orthonormal set of eigenvectors of $\tilde{T}$. If $\{E_t\}$ denotes the spectral family of $\tilde{T}$,

$$\begin{aligned} \|(\tilde{T} - \lambda)\varphi^{(m)}\|^2 &= \int_{-\infty}^{\infty} (\lambda - t)^2 \, d(E_t\varphi^{(m)}, \varphi^{(m)}) \\ &= \int_{-\infty}^{\Lambda+\delta} + \int_{\Lambda'-\delta}^{\infty} + \int_{\Lambda+\delta}^{\Lambda'-\delta} (\lambda - t)^2 \, d(E_t\varphi^{(m)}, \varphi^{(m)}) \\ &\geqq \{\tfrac{1}{2}(\Lambda' - \Lambda) - \delta\}^2 \left\{1 - \sum_{i=1}^{N} |(\varphi^{(m)}, \psi_i)|^2\right\} \\ &\quad + \sum_{i=1}^{N} (\lambda - \lambda_i)^2 |(\varphi^{(m)}, \psi_i)|^2. \end{aligned}$$

Since $(\varphi^{(m)}, \psi_i)\to 0$ as $m\to\infty$, $i=1, 2, \cdots, N$,

$$\Lambda' - \Lambda \leqq 2\delta + 2 \liminf_{m\to\infty} \|(\tilde{T} - \lambda)\varphi^{(m)}\|$$

and the result follows from (2.9) and (2.7) since δ is arbitrary.

If $\lambda \in (-\mu_j, \mu_j)$ we replace λ by μ_j in $\varphi^{(m)}$ above so that $f_m=0$, $v=\xi_j$ and $(\Omega-\Omega_0)Uv=(\Omega-\Omega_0)e_j$. Thus as above

$$\begin{aligned}\Lambda' - \Lambda &\leqq 2 \liminf_{m\to\infty} \|(\tilde{T} - \lambda)\varphi^{(m)}\| \\ &\leqq 2 \liminf_{m\to\infty} \|(\tilde{T} - \mu_j)\varphi^{(m)}\| + 2(\mu_j - \lambda)\end{aligned}$$

or

$$\Lambda' \leqq \mu_j + \liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \|(\Omega - \Omega_0)e_j\|_{A_m}.$$

Similarly, we obtain the lower bound for Λ by replacing λ by $-\mu_j$ in $\varphi^{(m)}$.

THEOREM 2. *Let A_m be as in Theorem* 1 *and suppose $n=1$. Let Ω_0 be a constant, real symmetric matrix with eigenvalues λ_1, λ_2, $\lambda_1\leqq\lambda_2$, and eigenvectors e_1, e_2.*

Then, if $\frac{1}{2}(\Lambda'+\Lambda) \in \complement(\lambda_1, \lambda_2)$,

$$\Lambda' - \Lambda \leqq 2 \liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \| \,|\Omega - \Omega_0|\, \|_{A_m}, \tag{2.10}$$

so that $\sigma_E \supset \complement(\lambda_1, \lambda_2)$ *if*

$$\liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \| \,|\Omega - \Omega_0|\, \|_{A_m} = 0. \tag{2.11}$$

If $\frac{1}{2}(\Lambda'+\Lambda) \in (\lambda_1, \lambda_2)$,

$$\Lambda' \leqq \lambda_2 + \liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \|(\Omega - \Omega_0)e_2\|_{A_m}, \tag{2.12}$$

$$\Lambda \geqq \lambda_1 - \liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \|(\Omega - \Omega_0)e_1\|_{A_m}. \tag{2.13}$$

PROOF. It is easy to show that

$$\Omega_0 - B\Omega_0 B = (\operatorname{Trace} \Omega_0)E_2 = (\lambda_1 + \lambda_2)E_2$$

and so

$$\tilde{\Omega}_0 = \tfrac{1}{2}(\Omega_0 + B\Omega_0 B) = \Omega_0 - \tfrac{1}{2}(\lambda_1 + \lambda_2)E_2. \tag{2.14}$$

Hence the eigenvalues of the B-selfadjoint matrix $\tilde{\Omega}_0$ are $\pm\tilde{\mu}_1$ where $\tilde{\mu}_1=\frac{1}{2}(\lambda_2-\lambda_1)$. The eigenvector $\tilde{e}_1$ of $\tilde{\Omega}_0$ corresponding to $\tilde{\mu}_1$ in e_2 and $B\tilde{e}_1=e_1$.

Clearly, (Λ, Λ') is a gap in σ_E if and only if $(\Lambda - \frac{1}{2}(\lambda_1+\lambda_2), \Lambda' - \frac{1}{2}(\lambda_1+\lambda_2))$ is a gap in the essential spectrum of the operator $T - \frac{1}{2}(\lambda_1+\lambda_2)E_2$. We now apply Theorem 1, with $n=1$, $j=1$ and Ω_0, Ω replaced by $\tilde{\Omega}_0$, $\Omega - \frac{1}{2}(\lambda_1+\lambda_2)E_2$ respectively, to $T - \frac{1}{2}(\lambda_1+\lambda_2)E_2$. Theorem 2 follows on replacing μ_j, e_j, Λ, Λ' in Theorem 1 by $\tilde{\mu}_1$, $\tilde{e}_1$, $\Lambda - \frac{1}{2}(\lambda_1+\lambda_2)$ and $\Lambda' - \frac{1}{2}(\lambda_1+\lambda_2)$ respectively and using (2.14).

In Theorem 6.10 of **[4]** it is shown that if $n=1$ and τ is assumed to be regular at a, then $\sigma_E \supset C(\lambda_1, \lambda_2)$ if

$$\lim_{x\to\infty} \frac{1}{X} \int_a^x |\Omega(t) - \Omega_0| \, dt = 0.$$

If in Theorem 2 we restrict our attention to matrices Ω which are such that $|\Omega(t)|$ is bounded in the intervals A_m then our condition (2.11) is clearly stronger than Weidmann's criterion. However without such a restriction, the two criteria are not compatible.

Theorem 2 is evidently a special case of Theorem 1 when Ω_0 is B-selfadjoint. It is natural to ask whether the B-selfadjointness of Ω_0 can be omitted in Theorem 1 to obtain an extension of Theorem 2 as it stands. However the method of proof does not allow for a general symmetric matrix Ω_0, for, by analogy with the proof of Theorem 2, this is seen to require an explicit and simple relationship between the eigenvalues and eigenvectors of $\tilde{\Omega}_0$ and Ω_0, and an almost random example will serve to eliminate this possibility for $n>1$. If however Ω_0 is assumed to be diagonal the method can be applied to obtain an extension of Theorem 2 for $n>1$. We have

THEOREM 3. *Let A_m be as in Theorem 1. Let Ω_0 be a constant real diagonal matrix $dg(\lambda_1, \cdots, \lambda_{2n})$ where the eigenvalues λ_j are not necessarily in increasing order.*

If $\lambda_j \geqq \lambda_{2n+1-j}$ and $\frac{1}{2}(\Lambda'+\Lambda) \in C(\lambda_{2n+1-j}, \lambda_j)$ then

$$\Lambda' - \Lambda \leqq 2 \liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \|\{|(\Omega - \Omega_0)\xi_j| + |(\Omega - \Omega_0)\xi_{2n+1-j}|\}\|_{A_m} \tag{2.15}$$

so that

$$\sigma_E \supset C \bigcap_{j=1}^{2n} (\lambda_{2n+1-j}, \lambda_j) \tag{2.16}$$

if

$$\liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \| \, |\Omega - \Omega_0| \, \|_{A_m} = 0.$$

If $\frac{1}{2}(\Lambda'+\Lambda) \in (\lambda_{2n+1-j}, \lambda_j)$ *then*

$$\Lambda' \leqq \lambda_j + \liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \|(\Omega - \Omega_0)\xi_j\|_{A_m}, \tag{2.17}$$

$$\Lambda \geqq \lambda_{2n+1-j} - \liminf_{m\to\infty} \frac{1}{(2a_m)^{1/2}} \|(\Omega - \Omega_0)\xi_{2n+1-j}\|_{A_m}. \tag{2.18}$$

PROOF. It is readily shown that

$$\Omega_0 - B\Omega_0 B = dg_{1\leq j\leq 2n}(\lambda_j + \lambda_{2n+1-j})$$

and

$$\tilde{\Omega}_0 = \tfrac{1}{2}(\Omega_0 + B\Omega_0 B) = \tfrac{1}{2}dg_{1\leq j\leq 2n}(\lambda_j - \lambda_{2n+1-j}).$$

Hence the eigenvalues of the B-selfadjoint matrix $\tilde{\Omega}_0$ are $\pm\tilde{\mu}_j, j=1, 2, \cdots, n$, where $\tilde{\mu}_j=\frac{1}{2}|\lambda_j-\lambda_{2n+1-j}|$. The $\tilde{\mu}_j$ are in decreasing order if the λ_j are in increasing order but may not be in any special order otherwise. If $\lambda_j \geqq \lambda_{2n+1-j}$, the eigenvectors $\tilde{e}_j$, $B\tilde{e}_j$ of $\tilde{\Omega}_0$ corresponding to $\tilde{\mu}_j$, $-\tilde{\mu}_j$ are ξ_j and $-\xi_{2n+1-j}$. We therefore have

$$\{\tilde{\Omega}_0 + \tfrac{1}{2}(\lambda_j + \lambda_{2n+1-j})\}v = \Omega_0 v \tag{2.19}$$

when $v=\tilde{e}_j$ and $B\tilde{e}_j$.

From these observations the theorem follows by applying Theorem 1 to $T-\frac{1}{2}(\lambda_j+\lambda_{2n+1-j})E_{2n}$ as in Theorem 2, replacing μ_j, e_j, Λ, Λ' in Theorem 1 by $\tilde{\mu}_j$, $\tilde{e}_j$, $\Lambda-\frac{1}{2}(\lambda_j+\lambda_{2n+1-j})$ and $\Lambda'-\frac{1}{2}(\lambda_j+\lambda_{2n+1-j})$ respectively.

Note that if, in Theorem 3, Ω_0 is a B-selfadjoint diagonal matrix with eigenvalues $\pm\mu_j$, $\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_n$ then, in view of (1.1),

$$\bigcap_{j=1}^{2n} (\lambda_{2n+1-j}, \lambda_j) = \bigcap_{j=1}^{n} (-\mu_j, \mu_j) = (-\mu_1, \mu_1).$$

It is interesting to note that (2.16) implies that σ_E contains the whole real line if $\bigcap_{j=1}^{2n} (\lambda_{2n+1-j},\lambda_j)=\varnothing$, a condition which can be fulfilled when $n=1$ only if $\lambda_1=\lambda_2$ and when Ω_0 is B-selfadjoint only if $\mu_1=0$. This suggests that the dependence of Theorem 3 on the ordering of the eigenvalues of Ω_0 along the diagonal, a feature which one is tempted to view with suspicion at first, is indeed of importance. For otherwise the same result would hold with the eigenvalues λ_j in increasing order and in this case (2.16) only guarantees that σ_E contains $\mathrm{C}(\lambda_n, \lambda_{n+1})$ which is not the whole of the real line if $\lambda_n\neq\lambda_{n+1}$. Of course, our results do not give any information about the nature of the spectrum inside $(\lambda_n, \lambda_{n+1})$.

In the final theorem below we assume that some of the eigenvalues of Ω are large in a certain sense at infinity.

THEOREM 4. *Let A_m be as in Theorem 1. In A_m suppose there is defined a real diagonal matrix-valued function $\Omega_m = dg(\lambda_1^{(m)}, \cdots, \lambda_{2n}^{(m)})$ which satisfies the following conditions:*

(i) *For some s, $1 \leqq s \leqq n$, $\lambda_s^{(m)}$, $\lambda_{2n+1-s}^{(m)}$ are continuously differentiable functions in A_m;*

(ii) *Either*

$$\min_{t \in A_m} \{\lambda_s^{(m)}(t), \lambda_{2n+1-s}^{(m)}(t)\} \to +\infty$$

or

$$\max_{t \in A_m} \{\lambda_s^{(m)}(t), \lambda_{2n+1-s}^{(m)}(t)\} \to -\infty \text{ as } m \to \infty;$$

(iii) *$|\lambda_s^{(m)}/\lambda_{2n+1-s}^{(m)}|$ is bounded as $m \to \infty$;*

(iv) *$\|(\lambda_i^{(m)})'/\lambda_i^{(m)}\|_{A_m} = o(a_m^{1/2})$ as $m \to \infty$ for $i = s, 2n+1-s$.*

Then

$$\Lambda' - \Lambda \tag{2.20}$$
$$\leqq 2 \liminf_{m \to \infty} \frac{1}{(2a_m)^{1/2}} \|\{|(\Omega - \Omega_m)\xi_s| + |(\Omega - \Omega_m)\xi_{2n+1-s}|\}\|_{A_m}.$$

Note that in $\Omega_m = dg(\lambda_1^{(m)}, \cdots, \lambda_{2n}^{(m)})$ the $\lambda_i^{(m)}$ are not assumed to be in increasing order.

PROOF. In $\varphi^{(m)}(\cdot, \lambda)$ we again take $\lambda = \frac{1}{2}(\Lambda' + \Lambda)$ but this time we choose f_m and v (now a C^{2n}-valued function $v^{(m)}$) to satisfy

$$(if'_m B + \Omega_m - \lambda)v^{(m)} = 0.$$

From

$$\det(if'_m B + \Omega_m - \lambda) = \prod_{i=1}^{n} \{(\lambda_i^{(m)} - \lambda)(\lambda_{2n+1-i}^{(m)} - \lambda) - (f'_m)^2\}$$

it follows that we may put

$$f_m(t, \lambda) = \int_{c_m - a_m}^{t} \{(\lambda_s^{(m)}(x) - \lambda)(\lambda_{2n+1-s}^{(m)}(x) - \lambda)\}^{1/2} \, dx, \qquad |t - c_m| \leqq a_m$$

and zero otherwise. From (ii), f_m is real for any fixed λ if m is sufficiently large. As for $v^{(m)} = \{v_i^{(m)}\}$, we put $v_i^{(m)} = 0$ for $i \neq s$, $2n+1-s$ and

$$v_s^{(m)} = \frac{(\lambda_{2n+1-s}^{(m)} - \lambda)^{1/2}}{|\lambda_s^{(m)} + \lambda_{2n+1-s}^{(m)} - 2\lambda|^{1/2}},$$

$$v_{2n+1-s}^{(m)} = \frac{(\lambda - \lambda_s^{(m)})^{1/2}}{|\lambda_s^{(m)} + \lambda_{2n+1-s}^{(m)} - 2\lambda|^{1/2}}.$$

As $m \to \infty$ we get, from (ii) and (iii),

$$|v^{(m)}| \sim 1, \qquad (v^{(m)})' = O\left(\frac{(\lambda_s^{(m)})'}{\lambda_s^{(m)}}\right) + O\left(\frac{(\lambda_{2n+1-s}^{(m)})'}{\lambda_{2n+1-s}^{(m)}}\right).$$

Hence, since

$$(\tau - \lambda)\varphi^{(m)} = (if'_m B + \Omega_m - \lambda)\varphi^{(m)} + k_m \exp(if_m)B(u_m(\cdot - c_m)v^{(m)})' + (\Omega - \Omega_m)\varphi^{(m)}$$

it follows from (iv) and (2.7) that, as $m \to \infty$,

$$\begin{aligned} \|(\tau - \lambda)\varphi^{(m)}\| &\leqq k_m \|u'_m(\cdot - c_m)\|_{A_m} + k_m \|(v^{(m)})'\|_{A_m} \\ &\quad + k_m \|\{|(\Omega - \Omega_m)\xi_s| + |(\Omega - \Omega_m)\xi_{2n+1-s}|\}\|_{A_m} \\ &= o(1) + k_m \|\{|(\Omega - \Omega_m)\xi_s| + |(\Omega - \Omega_m)\xi_{2n+1-s}|\}\|_{A_m}. \end{aligned}$$

The result now follows as in the proof of Theorem 1.

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