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NONCOINCIDENCE OF THE STRICT AND STRONG OPERATOR TOPOLOGIES

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ABSTRACT. Let E be an infinite-dimensional linear subspace of $C(S)$, the space of bounded continuous functions on a locally compact Hausdorff space S. If μ is a regular Borel measure on S, then each element of E may be regarded as a multiplication operator on $L^p(\mu)$ $(1 \leqq p < \infty)$. Our main result is that the strong operator topology this identification induces on E is properly weaker than the strict topology. For E the space of bounded analytic functions on a plane region G, and μ Lebesgue measure on G, this answers negatively a question raised by Rubel and Shields in [9]. In addition, our methods provide information about the absolutely p-summing properties of the strict topology on subspaces of $C(S)$, and the bounded weak star topology on conjugate Banach spaces.

1. **Introduction.** Let $C(S)$ denote the space of bounded, continuous, complex valued functions on a locally compact Hausdorff space S, and let $C_0(S)$ denote those functions in $C(S)$ which vanish at infinity. The *strict topology* β on $C(S)$ is the locally convex topology induced by the seminorms

$$f \to \|fk\|_\infty \qquad (f \in C(S)),$$

where k runs through $C_0(S)$ and $\|\cdot\|_\infty$ denotes the supremum norm. This topology was introduced in [1] by Buck who derived many of its funda, mental properties. In particular [1, Theorems 1 and 2]: β is complete-Hausdorff, and weaker than the norm topology; the norm and strictly bounded subsets of $C(S)$ coincide, and the β-dual of $C(S)$ can be identified with $M(S)$, the space of finite, regular Borel measures on S, where the pairing between the spaces is

$$(1.1) \qquad \langle f, \mu \rangle = \int f\, d\mu \qquad (f \in C(S), \mu \in M(S)).$$

Let μ be a (possibly infinite) regular Borel measure on S, as defined in [4, Section 52]. Note that built into this definition is the fact that $\mu(K) < \infty$

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for every compact subset K of S [**4**, p. 223]. For each f in $C(S)$ the equation $M_f g = fg$ $(g \in L^p(\mu))$ defines a bounded linear operator M_f on $L^p(\mu)$. It is not difficult to see from the fact that μ gives finite measure to compact sets that the linear map $f \to M_f$ is actually an isometry taking $C(S)$ into the space of all bounded linear operators on $L^p(\mu)$. Thus $C(S)$ inherits the *strong operator topology* $\sigma_p = \sigma_p(\mu)$, defined by the seminorms

$$f \to \left\{ \int |fg|^p \, d\mu \right\}^{1/p} \qquad (f \in C(S)), \tag{1.2}$$

where g runs through $L^p(\mu)$ [**3**, VI. 1.2, p. 475]. Note that σ_p is locally convex and Hausdorff.

Now $C(S)$ also acts on $C_0(S)$ by multiplication, and in this case the corresponding strong operator topology is the strict topology. In [**9**, 5.18(c), p. 274], Rubel and Shields asked if $\beta = \sigma_2(\mu)$ on the space $H^\infty(G)$, where μ is two-dimensional Lebesgue measure on G, and G supports nonconstant bounded analytic functions. In this case $H^\infty(G)$ is infinite dimensional [**9**, Section 2.3], so the question is answered in the negative by the following theorem, which is our main result.

THEOREM 1. *Let E be an infinite-dimensional linear subspace of $C(S)$, and suppose μ is a regular Borel measure on S. Then the strong operator topology $\sigma_p(\mu)$ induced on E by its action on $L^p(\mu)$ is properly weaker than the strict topology.*

The proof of this result occupies §3, and uses the notion of absolutely p-summing locally convex topologies, introduced in the next section. In §4 we comment briefly on the bounded strong operator topology and the bounded weak star topology.

2. **Absolutely p-summing topologies.** Let τ be a locally convex topology on a real or complex linear space E, and let $E' = E'_\tau$ denote the τ-dual of E (all τ-continuous linear functionals on E). For e' in E' and e in E we will write $\langle e, e' \rangle$ instead of $e'(e)$. A sequence (e_n) in E is called *τ-weakly p-summable* if $\sum |\langle e_n, e' \rangle|^p < \infty$ for all e' in E', and *τ-absolutely p-summable* if $\sum S(e_n)^p < \infty$ for every τ-continuous seminorm S on E $(1 \leq p < \infty)$. If every τ-weakly p-summable sequence is τ-absolutely p-summable, we say τ is *absolutely p-summing*. For example, the weak topology on a Banach space is absolutely p-summing for all p; but if the space is infinite-dimensional, then the Dvoretzky-Rogers theorem [**8**, Theorem 8, p. 350] asserts that the norm topology is absolutely p-summing for *no* p $(1 \leq p < \infty)$. Note that if τ is not absolutely p-summing, then neither is any stronger locally convex topology on E with the same continuous linear functionals.

The following lemma, which is an easy consequence of the Dvoretzky-Rogers theorem, is the key to our proof of Theorem 1. We note that the same idea has been used in [**6**, Example 2, p. 417].

LEMMA 1. *Let E be an infinite-dimensional normed space, and let F be a linear subspace of E' which norms E; that is,*

$$\|e\| = \sup\{|\langle e, f\rangle| : f \in F,\ \|f\| \leqq 1\} \tag{2.1}$$

for each e in E. Let τ denote the topology on E of uniform convergence on (norm) null sequences of F. Then τ is not absolutely p-summing ($1 \leqq p < \infty$).

PROOF. Since E is infinite-dimensional it follows from the Dvoretzky-Rogers theorem stated above that there is a sequence (e_n) in E which is weakly, but not absolutely, p-summable for the norm topology; that is, $\sum |\langle e_n, e'\rangle|^p < \infty$ for all e' in E', but $\sum \|e_n\|^p = \infty$. Since τ is weaker then the norm topology, every τ-continuous linear functional on E is norm continuous; hence (e_n) is τ-weakly p-summable. We claim that (e_n) is not τ-absolutely p-summable. For by (2.1) there exists f_n in F with $\|f_n\| \leqq 1$, and

$$|\langle e_n, f_n\rangle| > \|e_n\|/2^{1/p} \qquad (n = 1, 2, \cdots).$$

Let (a_n) be a sequence of nonnegative numbers such that $\lim a_n = 0$, and $\sum a_n^p \|e_n\|^p = \infty$, and let $g_n = a_n f_n$ ($n = 1, 2, \cdots$). Then $\lim \|g_n\| = 0$, so the equation $Se = \sup_n |\langle e, g_n\rangle|$ (e in E) defines a τ-continuous seminorm on E. But

$$\begin{aligned}\sum (Se_n)^p &\geqq \sum |\langle e_n, g_n\rangle|^p = \sum a_n^p |\langle e_n, f_n\rangle|^p \\ &\geqq \sum a_n^p \|e_n\|^p/2 = \infty,\end{aligned}$$

so τ is not absolutely p-summing. □

We will also need a result of J. B. Conway concerning factorization of subsets of $M(S)$. Recall that a subset H of $M(S)$ is called *tight* if for each $\varepsilon > 0$ there exists a compact subset K of S such that $|\mu|(S - K) < \varepsilon$ for each μ in H.

LEMMA 2 [**2**, THEOREM 2.2, P. 476]. *A bounded subset H of $M(S)$ is tight if and only if there is a bounded subset B of $M(S)$ and a function k in $C_0(S)$ such that $H = kB$.*

Here, of course, $kB = \{kb : b \in B\}$. We can now prove the main result of this section.

PROPOSITION 1. *Let E be an infinite-dimensional linear subspace of $C(S)$. Then the strict topology on E is not absolutely p-summing* ($1 \leqq p < \infty$).

PROOF. Since the strict dual of $C(S)$ is $M(S)$, where the spaces are paired by (1.1) [**1**, Theorem 2], it follows easily that the strict dual E'_β of E may be identified with the quotient space $M(S)/E^\circ$, via the pairing

$$\langle f, \alpha + E^\circ\rangle = \int f\,d\alpha \qquad (f \in E, \alpha \in M(S)),$$

where E° is the annihilator of E in $M(S)$ (see [**5**, Theorem 14.5, p. 120]). Moreover E'_β is a subspace of E', the norm dual of E, so it is a normed space.

We will need the fact that for each α in $M(S)$ the norm of the coset $\alpha+E^\circ$ viewed as a linear functional on E coincides with its norm as an element of $M(S)/E^\circ$. To see this, note that each e in E acts by integration as a linear functional on $M(S)$ of norm $\|e\|$, so the pairing (1.1) induces an isometric isomorphism of E into $M(S)'$. Standard Banach space theory now shows that the weak star closure $\bar{E}$ of E in $M(S)'$ is isometrically isomorphic to the dual of $M(S)/\bar{E}^\circ$, where $\bar{E}^\circ$ is the annihilator of $\bar{E}$ in $M(S)$. But $\bar{E}^\circ=E^\circ$, which proves our assertion.

Now the evaluation functionals $(\lambda_s : s \in S)$ defined by

$$\lambda_s(e) = e(s) \qquad (e \text{ in } E) \tag{2.2}$$

are strictly continuous and have norm $\leqq 1$, so E'_β norms E in the sense of Lemma 1; hence Lemma 1 shows that the topology τ of uniform convergence on norm null sequences in E'_β is not absolutely p-summing. Clearly τ is stronger than the weak topology induced on E by E'_β, so we will be finished if we prove that $\tau \leqq \beta$; for then $E'_\tau = E'_\beta$, hence β is not absolutely p-summing since τ is not.

To show that $\tau \leqq \beta$, suppose (e'_n) is a norm null sequence in E'_β. By the isometric identification of E'_β with $M(S)/E^\circ$ there exists a sequence (α_n) in $M(S)$ such that $\lim \|\alpha_n\|=0$, and for each n, $\langle e, e'_n\rangle = \int e\,d\alpha_n$ (e in E). It is easy to see that (the range of) (α_n) is tight, hence by Lemma 2 there is a bounded sequence (λ_n) in $M(S)$ and a function k in $C_0(S)$ such that $\alpha_n = k\lambda_n$ for all n. Thus for e in E,

$$\sup_n |\langle e, e'_n\rangle| = \sup_n \left| \int ek\,d\lambda_n \right| \leqq \|ek\|_\infty \sup_n \|\lambda_n\|.$$

Since the left side of this inequality is a typical τ-seminorm, and the right side is a β-continuous seminorm, we have $\tau \leqq \beta$. $\square$

Note that Proposition 1 shows in particular that on any infinite-dimensional linear subspace E of $C(S)$ the strict topology is not nuclear. This fact was first conjectured by Klaus D. Bierstedt for $E=H^\infty(D)$, D the open unit disc (private communication).

3. **Proof of Theorem 1.** For convenience we replace the function g in (1.2) by $|g|^p$. Thus the topology $\sigma_p=\sigma_p(\mu)$ is induced by the seminorms

$$S_g(f) = \left\{\int |f|^p\, g\, d\mu\right\}^{1/p} \qquad (f \in E), \tag{3.1}$$

where g runs through L^+, the class of nonnegative μ-integrable functions on S. Since the maximum of two L^+ functions is again in L^+, we see easily that the sets

$$\{f \in C(S): S_g f \leqq 1\} \qquad (g \in L^+) \tag{3.2}$$

form a base for the σ_p-neighborhoods of zero in E.

Now if $g \in L^+$, then it follows from the regularity of μ that $g\mu \in M(S)$. By the argument used in the proof of Lemma 2 [**2**, Theorem 2.2] with $H=\{g\mu\}$, there exists k in $C_0(S)$ and h in L^+ such that $g=k^p h$. Thus

$$S_g f \leqq \|fk\|_\infty \, \|h\|_1^{1/p} \qquad (f \in C(S)),$$

so $\sigma_p \leqq \beta$ on $C(S)$.

We complete the proof by showing that $\sigma_p \neq \beta$ on E whenever E is infinite-dimensional. If the strict dual E'_β of E is different from the σ_p-dual, then we are done; so suppose these duals coincide. We claim that in this case σ_p is absolutely p-summing; so again $\sigma_p \neq \beta$, this time by Proposition 1.

Recall that the norm on E'_β is the restriction of the E' norm. Suppose (e_n) is a weakly σ_p (hence β) p-summable sequence in E. Then, as in [**7**, §1.2.3, p. 22], the set

$$\left\{\sum_1^N a_n e_n : N = 1, 2, \cdots;\ \sum |a_n|^q \leqq 1\right\},$$

where $p^{-1}+q^{-1}=1$, is bounded in the weak topology induced on E by E'_β, hence strictly bounded by Mackey's theorem [**5**, §17.5, p. 155]. Since the strict and norm bounded subsets of E coincide [**1**, Theorem 1], we have

$$\sup\left|\sum_1^N a_n \langle e_n, e'\rangle\right| < \infty,$$

where the supremum is taken over all positive integers N, all sequences (a_n) in the unit ball of l^q, and all e' in the unit ball of E'_β. From this it follows easily that

$$\sup\left\{\sum_1^\infty |\langle e_n, e'\rangle|^p : e' \in E'_\beta,\ \|e'\| \leqq 1\right\} < \infty. \tag{3.3}$$

Now if S is a σ_p-continuous seminorm on E, then S is bounded on a set of the form (3.2), hence $S \leqq S_g$ for some g in L^+. Taking λ_s as in (2.2) we obtain:

$$\sum (Se_n)^p = \sum (S_g e_n)^p = \sum \int |e_n|^p \, g \, d\mu$$

$$= \int \left(\sum |\langle e_n, \lambda_s \rangle|^p\right) g(s) \, d\mu(s) \leqq \|g\|_1 \sup \sum |\langle e_n, e' \rangle|^p < \infty,$$

where the supremum in the last line is taken over all e' in E'_β with $\|e'\| \leqq 1$; a condition satisfied by each λ_s. That the supremum is finite follows from (3.3); hence σ_p is an absolutely p-summing topology on E, and $\sigma_p \neq \beta$. □

4. **The bounded weak star and bounded strong operator topologies.** Let E be a subspace of $C(S)$, and let $b\sigma_p = b\sigma_p(\mu)$ denote the *bounded strong operator topology* induced on E by its action on $L^p(\mu)$ (see [**3**, VI. 9.9, p. 512]); that is, the strongest topology on E agreeing with σ_p on norm bounded sets.

If X is a Banach space, then the *bounded weak star topology* on its dual X' is the strongest topology on X' agreeing with the weak star topology on bounded sets [**3**, V.3.3, p. 427]. According to the Banach-Dieudonné theorem [**3**, V.5.4], the bounded weak star topology on X' is just the topology of uniform convergence on null sequences of X. From this and Lemma 1 we get the following result, already noted by Lazar and Retheford for $X = c_0$ [**6**, Example 2, p. 417].

THEOREM 2. *If X is an infinite-dimensional Banach space, then the bounded weak star topology on X' is not absolutely p-summing. In particular, it is not nuclear.*

In [**10**, Theorem 2, p. 475] we showed that if E is a linear subspace of $C(S)$ whose unit ball is strictly compact, then E is the dual of the quotient Banach space $M(S)/E^\circ$, and the bounded weak star topology thus induced on E is just the strict topology. This quickly yields the following

THEOREM 3. *Suppose E is a linear subspace of $C(S)$ whose unit ball is strictly compact. Let μ be a regular Borel measure on S. Then $b\sigma_p(\mu) = \beta$.*

PROOF. By [**10**, Theorem 2] β is the strongest topology on E agreeing on bounded sets with the weak topology induced by $M(S)/E^\circ = E'_\beta$. The proof of Theorem 1 shows that $\sigma_p \leqq \beta$, so the unit ball of E is also σ_p-compact. But the topology σ_p is Hausdorff, so $\sigma_p = \beta$ on the unit ball of E, hence on every bounded set (since they are both vector topologies). Thus $b\sigma_p = \beta$. □

In particular note that if E is $H^\infty(G)$ and μ is Lebesgue measure on G, then the hypotheses of Theorem 3 are satisfied. Thus if G supports nonconstant bounded analytic functions, then $H^\infty(G)$ is infinite-dimensional; and the strict topology on it is the bounded strong operator topology, but not the strong operator topology.

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