NONCOINCIDENCE OF THE STRICT AND STRONG OPERATOR TOPOLOGIES

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Abstract. Let $E$ be an infinite-dimensional linear subspace of $C(S)$, the space of bounded continuous functions on a locally compact Hausdorff space $S$. If $\mu$ is a regular Borel measure on $S$, then each element of $E$ may be regarded as a multiplication operator on $L^p(\mu)$ $(1 \leq p < \infty)$. Our main result is that the strong operator topology this identification induces on $E$ is properly weaker than the strict topology. For $E$ the space of bounded analytic functions on a plane region $G$, and $\mu$ Lebesgue measure on $G$, this answers negatively a question raised by Rubel and Shields in [9]. In addition, our methods provide information about the absolutely $p$-summing properties of the strict topology on subspaces of $C(S)$, and the bounded weak star topology on conjugate Banach spaces.

1. Introduction. Let $C(S)$ denote the space of bounded, continuous, complex valued functions on a locally compact Hausdorff space $S$, and let $C_0(S)$ denote those functions in $C(S)$ which vanish at infinity. The strict topology $\beta$ on $C(S)$ is the locally convex topology induced by the seminorms

$$f \mapsto \|fk\|_\infty \quad (f \in C(S)),$$

where $k$ runs through $C_0(S)$ and $\|\cdot\|_\infty$ denotes the supremum norm. This topology was introduced in [1] by Buck who derived many of its fundamental properties. In particular [1, Theorems 1 and 2]: $\beta$ is complete-Hausdorff, and weaker than the norm topology; the norm and strictly bounded subsets of $C(S)$ coincide, and the $\beta$-dual of $C(S)$ can be identified with $M(S)$, the space of finite, regular Borel measures on $S$, where the pairing between the spaces is

$$\langle f, \mu \rangle = \int f \, d\mu \quad (f \in C(S), \mu \in M(S)).$$

Let $\mu$ be a (possibly infinite) regular Borel measure on $S$, as defined in [4, Section 52]. Note that built into this definition is the fact that $\mu(K) < \infty$
for every compact subset \( K \) of \( S \) \([4, \text{p. 223}].\) For each \( f \) in \( C(S) \) the equation 
\[
M_g f = fg \quad (g \in L^p(\mu))
\]
defines a bounded linear operator \( M_f \) on \( L^p(\mu). \) It is not difficult to see from the fact that \( \mu \) gives finite measure to compact sets that the linear map \( f \to M_f \) is actually an isometry taking \( C(S) \) into the space of all bounded linear operators on \( L^p(\mu). \) Thus \( C(S) \) inherits the strong operator topology \( \sigma_p(\mu), \) defined by the seminorms

\[
(1.2) \quad f \to \left( \int |fg|^p \, d\mu \right)^{1/p} \quad (f \in C(S)),
\]
where \( g \) runs through \( L^p(\mu) \) \([3, \text{VI. 1.2, p. 475}].\) Note that \( \sigma_p \) is locally convex and Hausdorff.

Now \( C(S) \) also acts on \( C_0(S) \) by multiplication, and in this case the corresponding strong operator topology is the strict topology. In \([9, 5.18(c), \text{p. 274}].\) Rubel and Shields asked if \( \beta = \sigma_p(\mu) \) on the space \( H^o(G), \) where \( \mu \) is two-dimensional Lebesgue measure on \( G, \) and \( G \) supports nonconstant bounded analytic functions. In this case \( H^o(G) \) is infinite dimensional \([9, \text{Section 2.3}].\) so the question is answered in the negative by the following theorem, which is our main result.

**Theorem 1.** Let \( E \) be an infinite-dimensional linear subspace of \( C(S), \) and suppose \( \mu \) is a regular Borel measure on \( S. \) Then the strong operator topology \( \sigma_p(\mu) \) induced on \( E \) by its action on \( L^p(\mu) \) is properly weaker than the strict topology.

The proof of this result occupies §3, and uses the notion of absolutely \( p \)-summing locally convex topologies, introduced in the next section. In §4 we comment briefly on the bounded strong operator topology and the bounded weak star topology.

2. Absolutely \( p \)-summing topologies. Let \( \tau \) be a locally convex topology on a real or complex linear space \( E, \) and let \( E' = E' \) denote the \( \tau \)-dual of \( E \) (all \( \tau \)-continuous linear functionals on \( E \)). For \( e' \) in \( E' \) and \( e \) in \( E \) we will write \( \langle e, e' \rangle \) instead of \( e'(e) \). A sequence \( (e_n) \) in \( E \) is called \( \tau \)-weakly \( p \)-summable if \( \sum |\langle e_n, e' \rangle|^p < \infty \) for all \( e' \) in \( E' \), and \( \tau \)-absolutely \( p \)-summable if \( \sum S(e_n)^p < \infty \) for every \( \tau \)-continuous seminorm \( S \) on \( E \) \((1 \leq p < \infty). \) If every \( \tau \)-weakly \( p \)-summable sequence is \( \tau \)-absolutely \( p \)-summable, we say \( \tau \) is absolutely \( p \)-summing. For example, the weak topology on a Banach space is absolutely \( p \)-summing for all \( p; \) but if the space is infinite-dimensional, then the Dvoretzky-Rogers theorem \([8, \text{Theorem 8, p. 350}].\) asserts that the norm topology is absolutely \( p \)-summing for no \( p \) \((1 \leq p < \infty). \) Note that if \( \tau \) is not absolutely \( p \)-summing, then neither is any stronger locally convex topology on \( E \) with the same continuous linear functionals.
The following lemma, which is an easy consequence of the Dvoretzky-Rogers theorem, is the key to our proof of Theorem 1. We note that the same idea has been used in [6, Example 2, p. 417].

**Lemma 1.** Let $E$ be an infinite-dimensional normed space, and let $F$ be a linear subspace of $E'$ which norms $E$; that is,

$$
\|e\| = \sup\{|\langle e, f \rangle| : f \in F, \|f\| \leq 1\}
$$

for each $e$ in $E$. Let $\tau$ denote the topology on $E$ of uniform convergence on (norm) null sequences of $F$. Then $\tau$ is not absolutely $p$-summing ($1 \leq p < \infty$).

**Proof.** Since $E$ is infinite-dimensional it follows from the Dvoretzky-Rogers theorem stated above that there is a sequence $(e_n)$ in $E$ which is weakly, but not absolutely, $\ell_p$-summable for the norm topology; that is, $\sum |\langle e_n, e' \rangle|^p < \infty$ for all $e'$ in $E'$, but $\sum \|e_n\|^p = \infty$. Since $\tau$ is weaker then the norm topology, every $\tau$-continuous linear functional on $E$ is norm continuous; hence $(e_n)$ is $\tau$-weakly $\ell_p$-summable. We claim that $(e_n)$ is not $\tau$-absolutely $\ell_p$-summable. For by (2.1) there exists $f_n$ in $F$ with $\|f_n\| \leq 1$, and

$$
|\langle e_n, f_n \rangle| > \|e_n\|^{2/p} \quad (n = 1, 2, \cdots).
$$

Let $(a_n)$ be a sequence of nonnegative numbers such that $\lim a_n = 0$, and $\sum a_n^p \|e_n\|^p = \infty$, and let $g_n = a_nf_n$ $(n=1, 2, \cdots)$. Then $\lim \|g_n\| = 0$, so the equation $Se = \sup_n |\langle e, g_n \rangle|$ ($e$ in $E$) defines a $\tau$-continuous seminorm on $E$. But

$$
\sum (Se_n)^p \geq \sum |\langle e_n, g_n \rangle|^p = \sum a_n^p |\langle e_n, f_n \rangle|^p \geq \sum a_n^p \|e_n\|^p/2 = \infty,
$$

so $\tau$ is not absolutely $p$-summing. 

We will also need a result of J. B. Conway concerning factorization of subsets of $M(S)$. Recall that a subset $H$ of $M(S)$ is called tight if for each $\varepsilon > 0$ there exists a compact subset $K$ of $S$ such that $|\mu|(S - K) < \varepsilon$ for each $\mu$ in $H$.

**Lemma 2 [2, Theorem 2.2, p. 476].** A bounded subset $H$ of $M(S)$ is tight if and only if there is a bounded subset $B$ of $M(S)$ and a function $k$ in $C_0(S)$ such that $H = kB$.

Here, of course, $kB = \{kb : b \in B\}$. We can now prove the main result of this section.

**Proposition 1.** Let $E$ be an infinite-dimensional linear subspace of $C(S)$. Then the strict topology on $E$ is not absolutely $p$-summing ($1 \leq p < \infty$).
PROOF. Since the strict dual of $C(S)$ is $M(S)$, where the spaces are paired by (1.1) [1, Theorem 2], it follows easily that the strict dual $E'_\beta$ of $E$ may be identified with the quotient space $M(S)/E^\circ$, via the pairing

$$\langle f, \alpha + E^\circ \rangle = \int f \, d\alpha \quad (f \in E, \alpha \in M(S)),$$

where $E^\circ$ is the annihilator of $E$ in $M(S)$ (see [5, Theorem 14.5, p. 120]). Moreover $E'_\beta$ is a subspace of $E'$, the norm dual of $E$, so it is a normed space.

We will need the fact that for each $\alpha$ in $M(S)$ the norm of the coset $\alpha + E^\circ$ viewed as a linear functional on $E$ coincides with its norm as an element of $M(S)/E^\circ$. To see this, note that each $e$ in $E$ acts by integration as a linear functional on $M(S)$ of norm $\|e\|$, so the pairing (1.1) induces an isometric isomorphism of $E$ into $M(S)'$. Standard Banach space theory now shows that the weak star closure $\hat{E}$ of $E$ in $M(S)'$ is isometrically isomorphic to the dual of $M(S)/E^\circ$, where $E^\circ$ is the annihilator of $E$ in $M(S)$. But $E^\circ = E^\circ$, which proves our assertion.

Now the evaluation functionals $(\lambda_s : s \in S)$ defined by

$$\lambda_s(e) = e(s) \quad (e \in E)$$

are strictly continuous and have norm $\leq 1$, so $E'_\beta$ norms $E$ in the sense of Lemma 1; hence Lemma 1 shows that the topology $\tau$ of uniform convergence on norm null sequences in $E'_\beta$ is not absolutely $p$-summing. Clearly $\tau$ is stronger than the weak topology induced on $E$ by $E'_\beta$, so we will be finished if we prove that $\tau \leq \beta$; for then $E'_\beta = E'_\beta$, hence $\beta$ is not absolutely $p$-summing since $\tau$ is not.

To show that $\tau \leq \beta$, suppose $(e'_n)$ is a norm null sequence in $E'_\beta$. By the isometric identification of $E'_\beta$ with $M(S)/E^\circ$ there exists a sequence $(\alpha_n)$ in $M(S)$ such that $\lim \|\alpha_n\| = 0$, and for each $n$, $\langle e' \alpha_n, e' \alpha_n \rangle = \int e \, d\alpha_n \quad (e \in E)$. It is easy to see that (the range of) $(\alpha_n)$ is tight, hence by Lemma 2 there is a bounded sequence $(\lambda_n)$ in $M(S)$ and a function $k$ in $C_0(S)$ such that $\alpha_n = k \lambda_n$ for all $n$. Thus for $e$ in $E$,

$$\sup_n |\langle e, e'_n \rangle| = \sup_n \left| \int e k \, d\lambda_n \right| \leq \|ek\|_\infty \sup_n \|\lambda_n\|.$$

Since the left side of this inequality is a typical $\tau$-seminorm, and the right side is a $\beta$-continuous seminorm, we have $\tau \leq \beta$. □

Note that Proposition 1 shows in particular that on any infinite-dimensional linear subspace $E$ of $C(S)$ the strict topology is not nuclear. This fact was first conjectured by Klaus D. Bierstedt for $E = H^\infty(D)$, $D$ the open unit disc (private communication).
3. Proof of Theorem 1. For convenience we replace the function $g$ in (1.2) by $|g|^p$. Thus the topology $\sigma_p = \sigma_p(\mu)$ is induced by the seminorms

$$S_p(f) = \left\{ \int |f|^p g \, d\mu \right\}^{1/p}$$

where $g$ runs through $L^+$, the class of nonnegative $\mu$-integrable functions on $S$. Since the maximum of two $L^+$ functions is again in $L^+$, we see easily that the sets

$$\{ f \in C(S) : S_p f \leq 1 \} \quad (g \in L^+)$$

form a base for the $\sigma_p$-neighborhoods of zero in $E$.

Now if $g \in L^+$, then it follows from the regularity of $\mu$ that $g\mu \in M(S)$. By the argument used in the proof of Lemma 2 [2, Theorem 2.2] with $H = \{ g\mu \}$, there exists $k \in C_0(S)$ and $h \in L^+$ such that $g = k^p h$. Thus

$$S_p f \leq \| f k \|_\infty \| h \|_1^{1/p} \quad (f \in C(S)),$$

so $\sigma_p \leq \beta$ on $C(S)$.

We complete the proof by showing that $\sigma_p \neq \beta$ on $E$ whenever $E$ is infinite-dimensional. If the strict dual $E'_\beta$ of $E$ is different from the $\sigma_p$-dual, then we are done; so suppose these duals coincide. We claim that in this case $\sigma_p$ is absolutely $p$-summing; so again $\sigma_p \neq \beta$, this time by Proposition 1.

Recall that the norm on $E'_\beta$ is the restriction of the $E'$ norm. Suppose $(e_n)$ is a weakly $\sigma_p$ (hence $\beta$) $p$-summable sequence in $E$. Then, as in [7, §1.2.3, p. 22], the set

$$\left\{ \sum_{1}^{N} a_n e_n : N = 1, 2, \ldots ; \sum |a_n|^q \leq 1 \right\},$$

where $p^{-1} + q^{-1} = 1$, is bounded in the weak topology induced on $E$ by $E'_\beta$, hence strictly bounded by Mackey’s theorem [5, §17.5, p. 155]. Since the strict and norm bounded subsets of $E$ coincide [1, Theorem 1], we have

$$\sup \left\{ \sum_{1}^{N} a_n \langle e_n, e' \rangle \right\} < \infty,$$

where the supremum is taken over all positive integers $N$, all sequences $(a_n)$ in the unit ball of $l^q$, and all $e'$ in the unit ball of $E'_\beta$. From this it follows easily that

$$\sup \left\{ \sum_{1}^{N} |\langle e_n, e' \rangle|^p : e' \in E'_\beta, \| e' \| \leq 1 \right\} < \infty.$$

Now if $S$ is a $\sigma_p$-continuous seminorm on $E$, then $S$ is bounded on a set of
the form (3.2), hence $S \subseteq S_g$ for some $g$ in $L^+$. Taking $\lambda_s$ as in (2.2) we obtain:

$$\sum (Se_n)^p = \sum (S_\sigma e_n)^p = \sum |e_n|^p g \, d\mu$$

$$= \int (\sum |\langle e_n, \lambda_s \rangle|^p) g(s) \, d\mu(s) \leq \|g\|_1 \sup \sum |\langle e_n, e' \rangle|^p < \infty,$$

where the supremum in the last line is taken over all $e'$ in $E'_p$ with $\|e'\| \leq 1$; a condition satisfied by each $\lambda_s$. That the supremum is finite follows from (3.3); hence $\sigma_p$ is an absolutely $p$-summing topology on $E$, and $\sigma_p \neq \beta$. □

4. The bounded weak star and bounded strong operator topologies. Let $E$ be a subspace of $C(S)$, and let $b\sigma_p = b\sigma_p(\mu)$ denote the bounded strong operator topology induced on $E$ by its action on $L^p(\mu)$ (see [3, VI. 9.9, p. 512]); that is, the strongest topology on $E$ agreeing with $\sigma_p$ on norm bounded sets.

If $X$ is a Banach space, then the bounded weak star topology on its dual $X'$ is the strongest topology on $X'$ agreeing with the weak star topology on bounded sets [3, V.3.3, p. 427]. According to the Banach-Dieudonné theorem [3, V.5.4], the bounded weak star topology on $X'$ is just the topology of uniform convergence on null sequences of $X$. From this and Lemma 1 we get the following result, already noted by Lazar and Retheford for $X = c_0$ [6, Example 2, p. 417].

**Theorem 2.** If $X$ is an infinite-dimensional Banach space, then the bounded weak star topology on $X'$ is not absolutely $p$-summing. In particular, it is not nuclear.

In [10, Theorem 2, p. 475] we showed that if $E$ is a linear subspace of $C(S)$ whose unit ball is strictly compact, then $E$ is the dual of the quotient Banach space $M(S)/E^0$, and the bounded weak star topology thus induced on $E$ is just the strict topology. This quickly yields the following

**Theorem 3.** Suppose $E$ is a linear subspace of $C(S)$ whose unit ball is strictly compact. Let $\mu$ be a regular Borel measure on $S$. Then $b\sigma_p(\mu) = \beta$.

**Proof.** By [10, Theorem 2] $\beta$ is the strongest topology on $E$ agreeing on bounded sets with the weak topology induced by $M(S)/E^0 = E'_p$. The proof of Theorem 1 shows that $\sigma_p \leq \beta$, so the unit ball of $E$ is also $\sigma_p$-compact. But the topology $\sigma_p$ is Hausdorff, so $\sigma_p = \beta$ on the unit ball of $E$, hence on every bounded set (since they are both vector topologies). Thus $b\sigma_p = \beta$. □

In particular note that if $E$ is $H^\infty(G)$ and $\mu$ is Lebesgue measure on $G$,
then the hypotheses of Theorem 3 are satisfied. Thus if \( G \) supports non-constant bounded analytic functions, then \( H^e(G) \) is infinite-dimensional; and the strict topology on it is the bounded strong operator topology, but not the strong operator topology.

**REFERENCES**


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