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ANOTHER LOCALLY CONNECTED HAUSDORFF CONTINUUM NOT CONNECTED BY ORDERED CONTINUA

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ABSTRACT. An example is given of a locally connected Hausdorff continuum which is not connected by ordered continua.

In 1960, S. Mardešić **[2]** gave an example of a locally connected Hausdorff continuum which is not connected by ordered continua (see also **[3]**). His example is somewhat complex, and it is the purpose of this paper to give a conceptually simpler example.

An ordered continuum is a totally ordered set $\{K, <\}$ such that K with the topology induced by the total order is compact and connected. Every closed subset of an ordered continuum has a first point and a last point in the order; they share many of the properties of metric arcs, and may be characterized as being Hausdorff continua with only two noncut points. A space X is said to be *connected by ordered continua* if for each two points x and y of X, there is an ordered continuum K with first and last points a and b and a continuous map $f: K \to X$ such that $f(a)=x$ and $f(b)=y$.

We use the notation (x, y), $(x, y]$ and $[x, y]$ for nondegenerate open, half open and closed intervals of ordered continua, or of the real numbers when x and y are numbers, and the notation $\langle x, y\rangle$, $\langle x, y, z\rangle$ for ordered pairs and triples. Following Kelley **[1]**, we let Ω_0 denote the set of countable ordinals, Ω denote the first uncountable ordinal and Ω' denote $\Omega_0 \cup \{\Omega\}$. We let L denote $\Omega_0 \times [0, 1) \cup \{\langle \Omega, 0\rangle\}$ with the order topology induced by the lexicographic order—$\langle p, q\rangle < \langle r, s\rangle$ if and only if $p<r$ or $p=r$ and $q<s$. Then L is an ordered continuum, sometimes called "the long interval", with first point $\langle 0, 0\rangle$ and last point $\langle \Omega, 0\rangle$.

1. **The spaces S_0 and S.** In the product space $C=L\times[0,1]\times[-1,1]$, let S_0 denote the *closure* of

$$\{\langle\langle \alpha, t\rangle, y, z\rangle \in C \mid z = \operatorname{Sin} \pi/(1-t)\}.$$

We suggest Figure 1 as a representation of S_0. S_0 is a continuum and is locally connected except at the points $\langle\langle \alpha, 0\rangle, y, z\rangle$ where $\alpha>0$.

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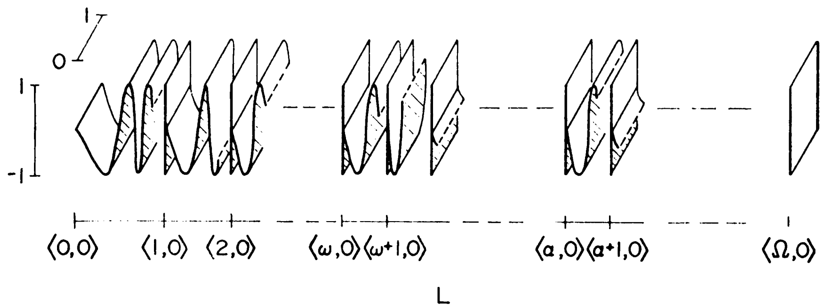

FIGURE 1

Let $\{\{\langle y_{\alpha,n}, z_{\alpha,n}\rangle\}_{n=1}^{\infty}\}_{\alpha\in\Omega_0}$ be a collection of countable dense subsets of $[0, 1]\times[-1, 1]$ with the property that $y_{\alpha,n}=y_{\bar{\alpha},\bar{n}}$ implies that $\alpha=\bar{\alpha}$ and $n=\bar{n}$. For α in Ω_0 and n a positive integer, we let $L_{\alpha,n}$ be the interval in C "parallel to L":

$$[\langle\alpha, 1-1/n\rangle, \langle\alpha+1, 0\rangle] \times \{y_{\alpha,n}\} \times \{z_{\alpha,n}\}.$$

(See Figure 2.) We let

$$S = S_0 \cup \bigcup \{L_{\alpha,n} \mid \alpha \in \Omega_0, n = 1, 2, \cdots\}.$$

Now S is also a continuum; the effect of adding the intervals $\{L_{\alpha,n}\}$ to S_0 is that S is locally connected at each point except the points $\langle\langle\alpha, 0\rangle, y, z\rangle$ where α is a *limit* ordinal. S is locally connected at the nonlimit ordinals, S_0 is not. However, for each number y_1 in $[0, 1]$, the "slice" $S_{y_1}=\{\langle\langle\alpha, t\rangle, y, z\rangle \in S | y=y_1\}$ of S contains at most one of the intervals $L_{\alpha,n}$, and is not locally connected at the points $\langle\langle\alpha, 0\rangle, y, z\rangle$, $\alpha>0$, except at an endpoint of perhaps that one interval.

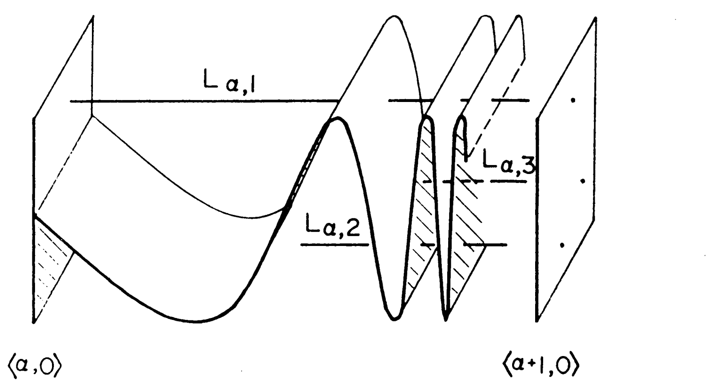

FIGURE 2

2. **The example** M**.** For each limit ordinal α in Ω' and each number y in $[0, 1]$, let $V_{\alpha,y}$ denote the "vertical" interval

$$V_{\alpha,y} = \{\langle\langle\alpha, 0\rangle, y, z\rangle \in S \mid -1 \leqq z \leqq 1\}.$$

Let M denote the upper semicontinuous decomposition of S whose only nondegenerate elements are the intervals $V_{\alpha,y}$ where α is a *limit* ordinal and $y \in [0, 1]$. (M is obtained from S by identifying to a point each of the intervals $V_{\alpha,y}$.) Let $q: S \to M$ be the quotient map of the decomposition. For each member P of M and open set O containing P, there is in C a product open set $R = U \times V \times W$, U open in L, V open in $[0, 1]$ and W open in $[-1, 1]$, such that $q(R \cap S)$ is a connected open set in M which contains P and is a subset of O. We omit discussion of the several special cases needed to establish this fact, but observe that with this and the fact that decompositions of continua yield continua, we have that M is a locally connected Hausdorff continuum.

3. M **is not connected by ordered continua.** Let E_Ω be $\{V_{\Omega,y} \in M \mid y \in [0, 1]\}$. We show that M is not connected by ordered continua between any point of $M - E_\Omega$ and any point of E_Ω. The crux of the argument is:

LEMMA. *If a subcontinuum H of $T = L \times [0, 1]$ contains only one point $\langle\langle\Omega, 0\rangle, y_1\rangle$ of $E = \{\langle\Omega, 0\rangle\} \times [0, 1]$ and a point $\langle\langle\alpha_0, t_0\rangle, y_0\rangle$ of $T - E$, then there is a nondegenerate interval $L' = (\langle\alpha', t'\rangle, \langle\Omega, 0\rangle]$ of L such that the intersection of H with $L' \times [0, 1]$ is $L' \times \{y_1\}$.*

PROOF. E is homeomorphic to $[0, 1]$, and there is a countable collection of open sets in T, $O_i = (\langle\alpha_i, t_i\rangle, \langle\Omega, 0\rangle] \times G_i$, G_i open in $[0, 1] - \{y_1\}$, $i = 1, 2, \cdots$, such that $E - H \subset \bigcup_{i=1}^{\infty} O_i \subset T - H$. Let $\langle\alpha', t'\rangle$ be the supremum in L of $\{\langle\alpha_0, t_0\rangle\} \cup \{\langle\alpha_i, t_i\rangle\}_{i=1}^{\infty}$ and $L' = (\langle\alpha', t'\rangle, \langle\Omega, 0\rangle]$. Because no countable sequence is cofinal in Ω_0, $\langle\alpha', t'\rangle$ is not $\langle\Omega, 0\rangle$ and L' is nondegenerate. The $\bigcup_{i=1}^{\infty} O_i$ contains all of $L' \times [0, 1]$ except $L' \times \{y_1\}$ and does not intersect H so that the intersection of H with $L' \times [0, 1]$ is a subset of $L' \times \{y_1\}$. Since H is connected, H must contain $L' \times \{y_1\}$ and the conclusion of the lemma follows.

Now suppose that $K = [a, b]$ is an ordered continuum and $f: K \to M$ is a continuous map such that $f(a) \in M - E_\Omega$ and $f(b) \in E_\Omega$. E_Ω is a closed subset of M and there is a first point, e, in K of $f^{-1}(E_\Omega)$. The map $p: M \to T$ ($T = L \times [0, 1]$), defined by

$$p(V_{\alpha,y}) = \langle\langle\alpha, 0\rangle, y\rangle \quad \text{for nondegenerate elements of } M,$$

$$p(\{\langle\langle\alpha, t\rangle, y, z\rangle\}) = \langle\langle\alpha, t\rangle, y\rangle \quad \text{for degenerate elements of } M,$$

is continuous, and $p(f([a, e]))$ is a continuum in T which contains only the point $p(f(e)) = \langle\langle\Omega, 0\rangle, y_1\rangle$ of E ($E = \{\langle\Omega, 0\rangle\} \times [0, 1]$) and a point $p(f(a))$

of $T-E$. We consider $\langle\alpha', t'\rangle$ and L' from the lemma above. For the number y_1, there is at most one interval L_{α,y_1} and we consider an ordinal $\bar{\alpha}$ less than Ω and greater than both α' and any α for which there is an L_{α,y_1}. There is a first point d of $[a, e]$ for which $p(f(d))=\langle\langle\bar{\alpha}+1, 0\rangle, y_1\rangle$ and a last point c of $[a, d]$ for which $p(f(c))=\langle\langle\bar{\alpha}, \frac{1}{2}\rangle, y_1\rangle$. Now $[c, d]$ is locally connected; hence $f([c, d])$ must be locally connected. But $f([c, d])$ must also be "the $y=y_1$ slice of M from $\langle\bar{\alpha}, \frac{1}{2}\rangle$ to $\langle\bar{\alpha}+1, 0\rangle$" (i.e. $\{\langle\langle\alpha, t\rangle, y, z\rangle\} \in M | y=y_1$ and $\langle\bar{\alpha}, \frac{1}{2}\rangle \leqq \langle\alpha, t\rangle \leqq \langle\bar{\alpha}+1, 0\rangle\}$) which is homeomorphic to the closure in the plane of the graph of $y=\sin(\pi/(1-x))$, $\frac{1}{2} \leqq x < 1$, and is not locally connected. This involves a contradiction and the proof is complete.

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