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MAP BORDISM OF MAPS

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ABSTRACT. Bordism is shown to distinguish homotopic maps.

1. **Introduction.** In this note, a *map* will mean a triple $\mathscr{f}=(f, X, Y)$ consisting of two topological spaces X and Y and a continuous function f from X to Y. A *morphism* $\alpha:\mathscr{f}\to\mathscr{f}'=(f', X', Y')$ will be a pair $\alpha=(a, a')$ of continuous functions $a:X\to X'$, $a': Y\to Y'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "a'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commutes. A *differentiable map* will mean a map $\Phi=(\phi, V, W)$ with V, W being compact manifolds with boundary and $\phi: V\to W$ a differentiable function with $\phi(\partial V)\subset\partial W$. The differentiable map Φ is *closed* if ∂V and ∂W are empty.

Being given a map $\mathscr{f}=(f, X, Y)$, a *bordism element* of $\mathscr{f}$ is an equivalence class of morphisms $\alpha:\Phi\to\mathscr{f}$ where Φ is a closed differentiable map. Two morphisms $\alpha_i=(a_i, a_i'):(\phi_i, M_i, N_i)\to\mathscr{f}$, $i=1, 2$, are equivalent if there is a morphism $\alpha=(a, a'):\Phi=(\phi, V, W)\to\mathscr{f}$ with Φ a differentiable map having $\partial V=M_1\cup M_2$, $\partial W=N_1\cup N_2$, $\phi/M_i=\phi_i$, $a/M_i=a_i$, $a'/N_i=a_i'$.

The set of bordism elements of $\mathscr{f}$ may be made into an abelian group $\mathfrak{N}_*(\mathscr{f})$ with the operation induced by the disjoint union; i.e. $[\alpha_1]+[\alpha_2]=[\alpha_1\cup\alpha_2]$, where $\alpha_1\cup\alpha_2=(a_1\cup a_2, a_1'\cup a_2')$ and $a_1\cup a_2: M_1\cup M_2\to X$, $a_1'\cup a_2'$: $N_1\cup N_2\to Y$ are the obvious maps on the disjoint unions. This will be called the map bordism group of $\mathscr{f}$.

The group $\mathfrak{N}_*(\mathscr{f})$ is a graded group, graded by the dimension, where if $\alpha:\Phi=(\phi, M, N)\to\mathscr{f}$, then the dimension of α is the dimension of N. There are subgroups of $\mathfrak{N}_*(\mathscr{f})$ obtained by considering only those bordism elements $\alpha:\Phi=(\phi, M, N)\to\mathscr{f}$ with dim $M=m$, dim $N=n$, which will be denoted $\mathfrak{N}_{m,n}(\mathscr{f})$. This is *not* a bigrading on $\mathfrak{N}_*(\mathscr{f})$, however.

Note. In [2], I was making the tacit assumption that the groups were bigraded. Only maps of dimension (m, n) were considered there and no erroneous statements appear in the paper.

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This is the obvious generalization for maps of the bordism groups of spaces defined by Atiyah [1], and, in fact, if $\boldsymbol{f}=(\phi, \emptyset, Y)$ is the empty map of the empty set into Y, then $\mathfrak{N}_n(\boldsymbol{f})\cong\mathfrak{N}_n(Y)$ is Atiyah's bordism group. Similarly, $\mathfrak{N}_{m,n}(\boldsymbol{f})\cong\mathfrak{N}_n(Y)$, which shows that the groups are not bigraded. For the map $\mathbf{1}=(1, p, p)$ which is the identity map on a single point, $\mathfrak{N}_{m,n}(\mathbf{1})=\mathfrak{N}_{m,n}$ is the bordism group of maps defined in [2].

Corresponding to a morphism $\alpha:\boldsymbol{f}\to\boldsymbol{f}'$, one has induced a homomorphism $\mathfrak{N}_*(\alpha):\mathfrak{N}_*(\boldsymbol{f})\to\mathfrak{N}_*(\boldsymbol{f}')$ by assigning to $\beta:\Phi\to\boldsymbol{f}$ the class of $\alpha\circ\beta$. Map bordism then gives a covariant functor on the category of maps and morphisms to the category of graded abelian groups and their morphisms. This functor has many of the properties of the usual bordism functor. For example, if $\alpha=(a, a')$ maps $(\phi\times 1: X\times[0,1], Y\times[0,1])$ into $\boldsymbol{f}'=(f', X', Y')$ and $\alpha_t=(a_t, a_t'):(\phi, X, Y)\to\boldsymbol{f}'$ is given by $a_t(x)=a(x,t)$, $a_t'(y)=a'(t,y)$, then $\mathfrak{N}_*(\alpha_0)=\mathfrak{N}_*(\alpha_1)$, i.e. homotopic morphisms induce the same homomorphism. The homomorphism $\mathfrak{N}_*(\alpha)$ also preserves the subgroups $\mathfrak{N}_{m,n}$; hence these are also covariant functors.

Unfortunately, it is very difficult to compute $\mathfrak{N}_*(\boldsymbol{f})$. Except for very restricted types of maps, I know no way to compute it.

The main result of this note is:

PROPOSITION. *Homotopic maps do not necessarily have isomorphic map bordism groups.*

To prove this, the map bordism groups of two maps will be computed. The maps are $f:\{-1,1\}\to[-1,1]$ given by inclusion, and $g:\{-1,1\}\to[-1,1]$ given by $g(t)=0$ for all t.

REMARK. The category of maps and their morphisms is a perfectly fine category in which to do homotopy theoretic analyses. However maps which are homotopic need not be homotopy equivalent in this category (e.g. f and g), and so homotopy theoretic methods may be used to distinguish homotopic maps.

2. **The map bordism of $\boldsymbol{f}$.** Let $\boldsymbol{f}=(f; \{-1,1\}, [-1,1])$ be the inclusion. Let $\alpha=(a, a'):\Phi=(\phi, M, N)\to\boldsymbol{f}$ be a representative for a bordism element.

Keeping a' fixed on $(a')^{-1}(\{-1,1\})$, one may make a' differentiable on $(a')^{-1}((-\frac{1}{2},\frac{1}{2}))$, by means of a small homotopy. This deformation is given by an equivalence, and so one may suppose a' is differentiable on $(a')^{-1}((-\frac{1}{2},\frac{1}{2}))$. By Sard's theorem, there is a regular value $r\in(-\frac{1}{2},\frac{1}{2})$ of this map, and let $N_r=(a')^{-1}(r)$. One may then find a neighborhood U of N_r in N and a diffeomorphism $\psi:N_r\times(-\varepsilon,\varepsilon)\to U$ so that $\psi(n,0)=n$ and such that $a'\psi(n,t)=t+r$.

The manifold N_r bounds, being in fact the boundary of $N_r'=(a')^{-1}([r,1])$.

Let $V=M\times[0,1]$, W the manifold obtained from $N\times[0,1]\cup N_r^+\times[-\varepsilon/2,\varepsilon/2]$ by identifying the points (n,t) of $N_r\times[-\varepsilon/2,\varepsilon/2]$ in $N_r^+\times[-\varepsilon/2,\varepsilon/2]$ with $(\psi(n,t);1)$ in $N\times[0,1]$, and then rounding corners, and let $\phi'=\phi\times 1:V\to W$. There is a retraction r of W on $N\times[0,1]\cup N_r^+\times\{0\}$, and let $A:V\to\{-1,1\}$ by $A(m,t)=a(m)$, $A':W\to[-1,1]$ by $A'(n,t)=a'(n)$ if $(n,t)\in N\times[0,1]$, $A'(n',0)=a'(n')$ if $(n',0)\in N_r^+\times\{0\}$, and $A'(x)=A'(rx)$ for all other x. Then $(A,A'):(\phi',V,W)\to\mathscr{f}$ is an equivalence of $\alpha:\Phi\to\mathscr{f}$ with a differentiable map $\beta:\Psi\to\mathscr{f}$ with $\beta=(b,b')$, $\Psi=(\eta,M',N')$, so that N' is the disjoint union of two submanifolds N_+ and N_- with $\eta(b^{-1}(-1))\subset N_-$, $\eta(b^{-1}(+1))\subset N_+$. By a homotopy one may deform b' so that $b'(N_+)=1$, $b'(N_-)=-1$.

Thus, if $i:\mathscr{j}=(1,\{-1,+1\},\{-1,1\})\to\mathscr{f}$ is the inclusion, this shows that $\mathfrak{N}_*(i)$ is epic. Since $\mathscr{j}$ is the disjoint union of two copies of the identity map on a point $\mathbf{1}=(1,p,p)$, one has an epimorphism

$$\mathfrak{N}_*(i):\mathfrak{N}_*(\mathbf{1})\oplus\mathfrak{N}_*(\mathbf{1})\to\mathfrak{N}_*(\mathscr{f}).$$

If $\Phi_i=(\phi_i,M_i,N_i)$, $i=1,2$, are two closed differentiable maps and $\alpha:\Phi_1\cup\Phi_2\to\mathscr{f}$ by sending M_1, N_1 into -1, M_2, N_2 into $+1$, and if α represents 0 in $\mathfrak{N}_*(\mathscr{f})$, then $\alpha:\Phi_1\cup\Phi_2\to\mathscr{f}$ is the boundary of $\beta=(b,b')$: $\Psi=(\psi,V,W)\to\mathscr{f}$. As above, one may split W into $W_r^+=(b')^{-1}([r,1])$, $W_r^-=(b')^{-1}([-1,r])$ where r is a regular value, with $W_r=(b')^{-1}(\{r\})$. One then has cobordisms of Φ_i with the map (ϕ,ϕ,W_r). Thus the kernel of $\mathfrak{N}_*(i)$ is a copy of $\mathfrak{N}_*$, which is a direct summand of $\mathfrak{N}_*(\mathbf{1})\oplus\mathfrak{N}_*(\mathbf{1})$, given by classes $\alpha:\Phi_1\cup\Phi_2\to\mathscr{f}$ where $\Phi_i=(\phi,\phi,N)$ and Φ_i is sent into the point map to $(-1)^i$. [If one augments $\mathfrak{N}_*(\mathbf{1})\oplus\mathfrak{N}_*(\mathbf{1})$ to $\mathfrak{N}_*\oplus 0$ by taking the bordism class of $(a')^{-1}(-1)$, one splits off this summand.]

Thus $\mathfrak{N}_*(i)$ induces an isomorphism

$$\mathfrak{N}_*(\mathbf{1})\oplus\mathfrak{N}_*(\mathbf{1})/\mathfrak{N}_*\xrightarrow{\cong}\mathfrak{N}_*(\mathscr{f})$$

or

$$\mathfrak{N}_{m,n}\oplus\mathfrak{N}_{m,n}/\mathfrak{N}_n\to\mathfrak{N}_{m,n}(\mathscr{f}).$$

Let $\Omega_\infty TBO_{\infty+n-m}$ be the space constructed in [2] with $\mathfrak{N}_{m,n}\cong\mathfrak{N}_n(\Omega_\infty TBO_{\infty+n-m})$; $\mathfrak{N}_n(\mathbf{1})$ is $\mathfrak{N}_n(\times_{m=-\infty}^{\infty}\Omega_\infty TBO_{\infty+n-m})$, where the product is given a weak topology as the limit of the finite products, and $\mathfrak{N}_n(\mathscr{f})$ is the group

$$\mathfrak{N}_n\left(\left(\bigtimes_{-\infty}^{\infty}\Omega_\infty TBO_{\infty+n-m}\right)\vee\left(\bigtimes_{-\infty}^{\infty}\Omega_\infty TBO_{\infty+n-m}\right)\right)$$

where $\vee$ denotes the union with a single common point.

3. **The map bordism of $\mathscr{g}$.** Let $\mathscr{g}=(g;\{-1,1\},[-1,1])$ with $g(t)=0$ for all t; $\mathscr{g}$ is homotopy equivalent in the category of maps and morphisms to the map $\mathscr{h}=(h,\{-1,1\},p)$ where p is a point, so $\mathfrak{N}_*(\mathscr{g})\simeq\mathfrak{N}_*(\mathscr{h})$.

A map bordism class in $\mathscr{h}$ is represented by a manifold N and maps $f_+ : M_+ \to N, f_- : M_- \to N$ where M_+ is sent to $+1$, M_- to -1, and N into p. The map f_+ is realized by a map $b_+ : N \to \mathop{\times}_{-\infty}^{\infty} \Omega_\infty TBO_{\infty+n-m}$ and f_- by $b_- : N \to \mathop{\times}_{-\infty}^{\infty} \Omega_\infty TBO_{\infty+n-m}$ (as in [2]), and so

$$\mathfrak{N}_*(\mathscr{g}) \cong \mathfrak{N}_*\left(\left(\mathop{\times}_{-\infty}^{\infty} \Omega_\infty TBO_{\infty+n-m}\right) \times \left(\mathop{\times}_{-\infty}^{\infty} \Omega_\infty TBO_{\infty+n-m}\right)\right).$$

Under the obvious morphisms $\alpha : \mathscr{f} \to \mathscr{h}$, $\beta : \mathscr{g} \to \mathscr{h}$ which restrict to the identities on $\{-1, 1\}$, the bordism homomorphism $\mathfrak{N}_*(\beta)$ is an isomorphism, while $\mathfrak{N}_*(\alpha)$ is induced by the inclusion of the wedge in the product.

The bordism of the wedge and product are not isomorphic, although for these spaces the bordism groups are infinite dimensional Z_2 vector spaces, and it is dissatisfying to say that the groups are not isomorphic on this basis.

For $n > m$, the space $\Omega_\infty TBO_{\infty+n-m}$ is connected and of finite type, with $\tilde{H}_*(\Omega_\infty TBO_{\infty+n-m})$ being nonzero in dimension $n-m$ and so for $n \geqq 2(n-m) > 0$, $\mathfrak{N}_n(\Omega_\infty TBO_{\infty+n-m} \vee \Omega_\infty TBO_{\infty+n-m})$ will not be isomorphic to $\mathfrak{N}_n(\Omega_\infty TBO_{\infty+n-m} \times \Omega_\infty TBO_{\infty+n-m})$. Thus, there exist values of n and m (with $n \geqq 2(n-m) > 0$) for which

$$\mathfrak{N}_{m,n}(\mathscr{f}) \not\cong \mathfrak{N}_{m,n}(\mathscr{g})$$

even as abstract groups.

Note. $\mathscr{f}$ may be considered as the map given by the mapping cylinder of $\mathscr{h}$. Hence, a map may not be replaced by its mapping cylinder in studying homotopy theoretic functors on this category.

References

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