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EXTREMAL PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS[1]

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ABSTRACT. An extremal problem for the set of probability measures on the unit circle is formulated and solved. The result contains as special cases many of the published results concerning extremal problems for analytic functions with positive real part, starlike functions, and other classes of functions associated with integral formulas.

There are several important classes of analytic functions that are given by integral formulas involving the set P of probability measures on the unit circle. For example the Herglotz formula

$$f(z) = \int (1 + \zeta z)/(1 - \zeta z)\, d\mu(\zeta) \qquad (|z| < 1) \tag{1}$$

yields, as μ varies over P, all analytic functions f with $f(0)=1$ and with $\operatorname{Re} f(z)>0$ for $|z|<1$, and the formula

$$g(z) = z \exp \int - 2 \log(1 - \zeta z)\, d\mu(\zeta) \qquad (|z| < 1) \tag{2}$$

gives the class of starlike functions. Many authors have studied various extremal problems for such classes by employing variational formulas, often quite complicated, for the functions of the class being considered. (See [1], [3], [4], [5], [6], [7], and the sample problems in the next paragraph.) There is a simple striking feature common to the solutions of all these problems: Any extremizing function is associated with a measure consisting entirely of finitely many point masses. Thus any solution of an extremal problem (of the type considered) for the functions (1) with positive real part has the form

$$f(z) = \sum_{k=1}^{m} \mu_k (1 + \zeta_k z)/(1 - \zeta_k z) \qquad \left(|\zeta_k| = 1, \mu_k > 0, \sum_{k=1}^{m} \mu_k = 1\right), \tag{3}$$

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any solution to a problem for the starlike functions (2) has the form

$$g(z) = z \Big/ \prod_{k=1}^{m} (1 - \zeta_k z)^{2\mu_k}, \tag{4}$$

etc. This suggests the possibility of a single extremal problem for the set of measures P sufficiently general to encompass all problems of the type alluded to above, and such that the extremizing measures are always composed of finitely many point masses. We shall presently formulate such a problem and solve it using as a basic tool a simple variational technique for measures due essentially to Ulin [**8**]. Thus one can handle simultaneously all the function classes discussed in this connection and frequently eliminate the need for variational formulas tailored to the various classes. We discuss the actual number of point masses in the next-to-last paragraph of the paper.

To see what form our extremal problem should take we now consider a couple of examples from the literature.

Problem 1. (See [7].) Let $F(w_1, \cdots, w_n)$ be analytic for $|w_j| \leqq 2$ $(1 \leqq j \leqq n)$. Find $\max_f \operatorname{Re} F(p_1, \cdots, p_n)$ over the class (1), where $f(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$. We rephrase the problem in terms of the set of measures P as follows:

$$\begin{aligned} f(z) &= \int [(1 + \zeta z)/(1 - \zeta z)]\, d\mu(\zeta) = \int \left(1 + 2\sum_{k=1}^{\infty} \zeta^k z^k\right) d\mu(\zeta) \\ &= 1 + 2 \sum_{k=1}^{\infty} z^k \int \zeta^k\, d\mu(\zeta). \end{aligned}$$

Hence $p_k = 2\int \zeta^k\, d\mu(\zeta)$ and

$$\max_f \operatorname{Re} F(p_1, \cdots, p_n) = \max_\mu \operatorname{Re} F(I_1, \cdots, I_n),$$

where

$$I_j = \int \phi_j(\zeta)\, d\mu(\zeta), \quad \phi_j(\zeta) = 2\zeta^j \qquad (1 \leqq j \leqq n). \tag{5}$$

More generally, suppose we seek $\max_f \operatorname{Re} F(f(z), f'(z), \cdots, f^{(n-1)}(z))$ over the class (1) where $|z| < 1$ and $F(w_1, \cdots, w_n)$ is analytic for $\operatorname{Re} w_1 > 0$, $|w_j| < \infty$ $(2 \leqq j \leqq n)$. Then

$$\max_f \operatorname{Re} F(f(z), f'(z), \cdots, f^{(n-1)}(z)) = \max_\mu F(I_1, \cdots, I_n)$$

where

$$I_j = \int \phi_j(\zeta)\, d\mu(\zeta), \quad \phi_j(\zeta) = D_z^{j-1}(1 + \zeta z)/(1 - \zeta z) \qquad (1 \leqq j \leqq n). \tag{6}$$

Problem 2. (See [**1**].) Let $0<|z|<1$ and find $\max_g \operatorname{Re} F(\log[g(z)/z])$ over the class (2) where F is entire. Here $n=1$ and

$$\max_g \operatorname{Re} F(\log[g(z)/z]) = \max_\mu \operatorname{Re} F(I_1),$$

where

$$I_1 = \int \phi_1(\zeta)\, d\mu(\zeta), \qquad \phi_1(\zeta) = -2\log(1-\zeta z). \tag{7}$$

For this choice of ϕ_1 there are several interesting choices of F. (See [**1**].)

Before we turn to our theorem on extremal problems for P, we make some observations about the above examples. First, the functions ϕ_j are all analytic in the closed unit disk. Secondly, the existence of all the above maxima and the maximum in the theorem below is clear from several points of view. For instance with the relative weak-star topology, P is a compact subset of the conjugate space of the Banach space of continuous functions on the unit circle, and the map $\mu \to F(I_1, \cdots, I_n)$ is continuous in this topology. Finally, the linear independence condition of the theorem is easily verified for our examples.

Theorem. *Let $\phi_1, \cdots, \phi_n$ be analytic in the closed unit disk $\{w: |w| \leqq 1\}$, and let $1, \phi_1, \cdots, \phi_n$ be linearly independent. Let P be the set of probability measures on the Borel subsets of the unit circle $\{\zeta: |\zeta|=1\}$. For each $\mu \in P$ let $I(\mu)$ be the vector (in C^n) defined by*

$$I(\mu) = \left(\int \phi_1(\zeta)\, d\mu(\zeta), \cdots, \int \phi_n(\zeta)\, d\mu(\zeta)\right). \tag{8}$$

Let F be analytic and nonconstant on $\{I(\mu): \mu \in P\}$. Then any $\nu \in P$ such that

$$\operatorname{Re} F(I(\nu)) = \max\{\operatorname{Re} F(I(\mu)) : \mu \in P\} \tag{9}$$

consists of finitely many point masses on the unit circle.

Proof. We divide the proof into four parts, Part 1 being quite simple.

Part 1. For each $\mu \in P$ we introduce the notation

$$\mu^{(k)} = \int \zeta^k\, d\mu(\zeta) \qquad (k = 0, 1, 2, \cdots). \tag{10}$$

Then for any $\mu \in P$ and any w with $|w| \leqq 1$, $\exists \mu_w \in P$ such that

$$\mu_w^{(k)} = \mu^{(k)} w^k \qquad (k = 0, 1, 2, \cdots). \tag{11}$$

Proof of Part 1. Since $\operatorname{Re} \int (1+wz\zeta)/(1-wz\zeta)\, d\mu(\zeta) > 0$ for $|z|<1$, the Herglotz theorem gives $\mu_w \in P$ such that

$$\int (1 + wz\zeta)/(1 - wz\zeta)\, d\mu(\zeta) = \int (1 + z\zeta)/(1 - z\zeta)\, d\mu_w(\zeta) \qquad (|z| < 1).$$

The desired result follows by subtracting 1, expanding both integrands into series, and integrating termwise. Another proof, due to Ken Binmore, consists of exhibiting explicitly a measure $\lambda_w \in P$ such that $\lambda_w^{(k)}=w^k$. Then $\mu_w=\mu * \lambda_w$. Finally, we note that for any $\mu \in P$, μ_0 is normalized Lebesgue measure and $\mu_1=\mu$.

Part 2. If ν is maximal as in (9), then not all the first partial derivatives $F_j(I(\nu))$ vanish $(1\leqq j\leqq n)$.

PROOF OF PART 2. Here we reproduce the argument of Kirwan **[2]** with appropriate modifications. Since ϕ_j $(1\leqq j\leqq n)$ is analytic in the closed unit disk, $\exists R>1$ for which we can write

$$\phi_j(w) = \sum_{k=0}^{\infty} a_{k,j}w^k \qquad (|w| < R,\ 1 \leqq j \leqq n). \tag{12}$$

Since $|\nu^{(k)}|\leqq 1$ $(k=0, 1, 2, \cdots)$ we can define the analytic functions ψ_j $(1\leqq j\leqq n)$ by

$$\psi_j(w) = \sum_{k=0}^{\infty} a_{k,j}\nu^{(k)}w^k \qquad (|w| < R,\ 1 \leqq j \leqq n). \tag{13}$$

Then, using Part 1, we note that, for $|w|\leqq 1$,

$$\psi_j(w) = \sum_{k=0}^{\infty} a_{k,j}\nu_w^{(k)} = \int \phi_j(\zeta)\, d\nu_w(\zeta).$$

Hence

$$(\psi_1(w), \cdots, \psi_n(w)) = I(\nu_w) \qquad (|w| \leqq 1). \tag{14}$$

From this and the fact that F is analytic on an open set containing $\{I(\mu):\mu \in P\}$ it follows that the function

$$h(w) = F(\psi_1(w), \cdots, \psi_n(w)) \tag{15}$$

is analytic on the closed unit disk. Let us also note that

$$(\psi_1(1), \cdots, \psi_n(1)) = I(\nu), \qquad h(1) = F(I(\nu)). \tag{16}$$

Now we suppose, contrary to the assertion of Part 2, that $F_j(I(\nu))=0$ $(1\leqq j\leqq n)$. Then (15) and (16) yield $h'(1)=0$. Consequently the Taylor expansion of h about 1 has the form

$$h(w) = F(I(\nu)) + \sum_{k=2}^{\infty} b_k(w-1)^k \qquad (|w-1| < \delta).$$

Thus, (14) and (15) give

$$F(I(\nu_w)) = F(I(\nu)) + \sum_{k=2}^{\infty} b_k(w-1)^k \qquad (|w| \leqq 1,\ |w-1| < \delta). \tag{17}$$

By (9) we obtain

$$\text{(18)} \qquad \operatorname{Re} \sum_{k=2}^{\infty} b_k(w-1)^k \leqq 0 \qquad (|w| \leqq 1, |w-1| < \delta).$$

It follows easily from (18) that

$$\text{(19)} \qquad b_k = 0 \qquad (k \geqq 2).$$

Indeed, for any θ satisfying $\pi/2<\theta<3\pi/2$, there exists $r>0$ so small that $w-1=re^{i\theta}$ implies $|w|\leqq 1$ and $|w-1|<\delta$. For such w, (18) becomes $\operatorname{Re}[r^2e^{2i\theta}b_2+o(r^2)]\leqq 0$. Therefore, $\operatorname{Re} e^{2i\theta}b_2\leqq 0$ $(\pi<2\theta<3\pi)$. From this we conclude that $b_2=0$, and similarly all $b_k=0$. Now (19) reduces (17) to

$$\text{(20)} \qquad F(I(\nu_w)) = F(I(\nu)) \qquad (|w| \leqq 1, |w-1| < \delta).$$

Since $F(I(\nu_w))$ is analytic for $|w|\leqq 1$, we obtain the stronger equation

$$\text{(21)} \qquad F(I(\nu_w)) = F(I(\nu)) \qquad (|w| \leqq 1).$$

In particular $F(I(\nu_0))=F(I(\nu))$. But then we obtain a contradiction to the nonconstancy of F. Indeed, for any $\mu \in P$,

$$\text{(22)} \qquad \operatorname{Re} F(I(\mu_0)) = \operatorname{Re} F(I(\nu_0)) = \operatorname{Re} F(I(\nu)) \geqq \operatorname{Re} F(I(\mu_w)).$$

Since $F(I(\mu_w))$, like $F(I(\nu_w))$, is analytic for $|w|\leqq 1$, (22) implies that $F(I(\mu_w))$ is constant for $|w|\leqq 1$. Therefore, $F(I(\mu_0))=F(I(\mu_1))=F(I(\mu))$, so that F is constant on $\{I(\mu):\mu \in P\}$.

Part 3. There are at most finitely many points v on the unit circle for which $\nu(\{v\})>0$.

PROOF OF PART 3. Here and in the concluding Part 4 we use the variational method of Ulin [**8**]. We suppose $|v|=1$ and $\nu(\{v\})>0$. Let $0<\varepsilon<\nu(\{v\})$, let $|w|=1$, $w\neq v$, and let μ be the measure obtained from ν by removing mass ε from $\{v\}$ and placing it on $\{w\}$. In symbols $\mu=\nu-\varepsilon\delta_v+\varepsilon\delta_w$, where δ_v and δ_w are unit point masses at v and w respectively. Then $\mu \in P$ and

$$\int \phi_j(\zeta)\, d\mu(\zeta) - \int \phi_j(\zeta)\, d\nu(\zeta) = \varepsilon[\phi_j(w) - \phi_j(v)] \qquad (1 \leqq j \leqq n).$$

Since F is analytic at $I(\nu)$ we have for ε sufficiently small the Taylor expansion

$$\text{(23)} \qquad F(I(\mu)) = F(I(\nu)) + \varepsilon \sum_{j=1}^{n} F_j(I(\nu))[\phi_j(w) - \phi_j(v)] + o(\varepsilon).$$

From the maximality property (9) of ν we conclude

$$\mathrm{Re} \sum_{j=1}^{n} F_j(I(\nu))[\phi_j(w) - \phi_j(v)] \leqq 0,$$

$$\mathrm{Re} \sum_{j=1}^{n} F_j(I(\nu))\phi_j(v) \geqq \mathrm{Re} \sum_{j=1}^{n} F_j(I(\nu))\phi_j(w).$$

Since w was arbitrary,

$$\text{(24)} \qquad \mathrm{Re} \sum_{j=1}^{n} F_j(I(\nu))\phi_j(v) = \max\left\{\mathrm{Re} \sum_{j=1}^{n} F_j(I(\nu))\phi_j(w) : |w| = 1\right\}.$$

If the assertion of Part 3 is false, (24) holds for infinitely many v. Since the ϕ_j $(1 \leqq j \leqq n)$ are analytic in the closed unit disk, it follows by a standard argument that (24) then holds for all v on the unit circle. But this in turn implies that $\sum_{j=1}^{n} F_j(I(\nu))\phi_j(w)$ is constant for $|w| \leqq 1$. By Part 2, not all $F_j(I(\nu))$ equal zero $(1 \leqq j \leqq n)$. Hence we have contradicted the linear independence of $1, \phi_1, \cdots, \phi_n$.

Part 4. There are no points v on the unit circle such that $\nu(\{v\}) = 0$ and $\nu(N_v) > 0$ for every open neighborhood N_v of v in the unit circle.

PROOF OF PART 4. Suppose there is a point v with the above two properties. Let $|w| = 1$, $w \neq v$. Let N_v be any open neighborhood of v such that $w \notin N_v$, and let $\varepsilon = \nu(N_v)$. Let μ be the measure obtained from ν by transferring the mass ε from N_v to $\{w\}$. Then

$$\int \phi_j(\zeta)\, d\mu(\zeta) - \int \phi_j(\zeta)\, d\nu(\zeta) = \varepsilon\phi_j(w) - \int_{N_v} \phi_j(\zeta)\, d\nu(\zeta)$$

$$= \varepsilon[\phi_j(w) - \phi_j(v)] + \int_{N_v} [\phi_j(v) - \phi_j(\zeta)]\, d\nu(\zeta) \qquad (1 \leqq j \leqq n).$$

The assumption $\nu(\{v\}) = 0$ implies that ε can be made arbitrarily small by a suitable choice of N_v. Similarly $\sup\{|\phi_j(v) - \phi_j(\zeta)| : \zeta \in N_v\}$ can be made arbitrarily small by the continuity of ϕ_j $(1 \leqq j \leqq n)$. Hence we can write

$$\int_{N_v} [\phi_j(v) - \phi_j(\zeta)]\, d\nu(\zeta) = o(\varepsilon)$$

with the understanding that $o(\varepsilon)/\varepsilon$ tends to 0 for suitable ε tending to 0. It follows that we can again write the Taylor expansion (23) for suitable, small $\varepsilon > 0$. Moreover (24) follows as before, and finally we conclude that there are at most finitely many points v with the two properties of Part 4. But then, if v is any such point, we can find a neighborhood N_v of v excluding all other points of the type described in Parts 3 and 4. In other words each point of $N_v \backslash \{v\}$ contains a neighborhood with no ν-mass.

Hence $N_v \setminus \{v\}$ itself contains no mass, a contradiction. The theorem now follows.

We remark that it is frequently possible to state an upper bound for the number of points v ($|v|=1$) such that $\nu(\{v\})>0$. Equation (24), which must be satisfied by such points, is then useful. For example in our illustrative Problem 1, where $\phi_j(v)=2v^j$ $(1\leqq j\leqq n)$, the left side of (24) becomes a trigonometric polynomial of order at most n. Such a function achieves its maximum at most n times, so that n is the required upper bound. In Problem 2 the left side of (24) has the form $\operatorname{Re} \lambda \log(1-vz)$, where $\lambda\neq 0$ and $0<|z|<1$. The function $\lambda \log(1-vz)$ maps the unit circle $|v|=1$ onto a convex analytic curve in one-to-one fashion. Each line of support of this curve meets the curve in exactly one point. Consequently there is exactly one point v for which $\operatorname{Re} \lambda \log(1-vz)$ achieves its maximum. Thus the extremal measure ν is a unit point mass, and the extremal function is a Koebe function.

Finally we observe that the single reason for our hypotheses that the probability space be the unit circle, that the ϕ_j be analytic, and that 1, $\phi_1, \cdots, \phi_n$ be linearly independent was to ensure that (24) hold for only finitely many points v. If this fact is known or assumed, a much simpler and more general theorem, based only on Parts 3 and 4, becomes possible.

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