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ON SEMIBOUNDED DIFFERENTIAL OPERATORS

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ABSTRACT. It is shown that regular ordinary differential operators have a semibounded spectrum. The proof requires fewer prerequisites than other proofs found in the literature and also yields estimates on the lower bound of the spectrum.

The purpose of this note is to provide a short and relatively elementary proof of the following theorem.

THEOREM. *Let $p_0(x), p_1(x), \cdots, p_n(x)$ be a set of real functions on $[0, 1]$, such that $p_k(x) \in C^k[0, 1]$ for $k=0, 1, \cdots, n$. Let $\{a_k\}$ denote their lower bounds such that $p_k(x) \geqq a_k$. In particular $p_n(x)$ is to be positive so that $p_n(x) \geqq a_n > 0$. Define the formally selfadjoint differential operator l by*

$$(1) \qquad l(u) = \sum_{k=0}^{n} (-1)^k (p_k(x) u^{(k)})^{(k)}.$$

l is defined on a suitable domain in $L_2[0, 1]$.

We now consider the boundary conditions

$$(2) \qquad \sum_{j=0}^{2n-1} \alpha_{ij} u^{(j)}(0) + \sum_{j=0}^{2n-1} \beta_{ij} u^{(j)}(1) = 0, \qquad i = 1, 2, \cdots, 2n.$$

We assume that for all functions $u, v \in C^{2n}[0, 1]$ that also satisfy (2) we have

$$(3) \qquad (lu, v) - (u, lv) = \int_0^1 [(lu)\bar{v} - u(\overline{lv})]\, dx = 0.$$

Let L_1 denote the regular selfadjoint operator generated by l and the boundary conditions (2) on the Hilbert space $L_2[0, 1]$. The spectrum of L_1 is bounded below.

For other proofs of this theorem see [1], [2], [3]. The proof of the theorem will be based on the following lemma.

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LEMMA. *Let l be as in the above theorem. Consider the boundary conditions*

$$u^{(j)}(0) = 0, \qquad u^{(j)}(1) = 0, \qquad j = 0, 1, \cdots, n-1. \tag{4}$$

Denote by L_0 the regular selfadjoint operator generated by l and the boundary conditions (4). *The spectrum of L_0 is bounded below.*

PROOF OF THE LEMMA. By integration by parts, use of the boundary conditions (4) and the fact that all $p_k(x) \geqq a_k$ we have

$$(L_0 u, u) = \int_0^1 \sum_{k=0}^{n} p_k \, |u^{(k)}|^2 \, dx \geqq \int_0^1 \sum_{k=0}^{n} a_k \, |u^{(k)}|^2 \, dx. \tag{5}$$

In order for u to be in the domain of L_0 it must have at least n continuous derivatives. We can therefore expand u in a Fourier series that can be differentiated term by term n times. Then

$$u = \sum_{m=1}^{\infty} b_m \sin m\pi x \tag{6}$$

and inserting the above in (5) we have

$$(L_0 u, u) \geqq \tfrac{1}{2} \sum_{m=1}^{\infty} |b_m|^2 \sum_{k=0}^{n} a_k (m\pi)^{2k}. \tag{7}$$

We now define the polynomial $q(x) = \sum_{k=0}^{n} a_k x^{2k}$ and find, using (7), that

$$(L_0 u, u) \geqq \tfrac{1}{2} \min_m q(m\pi) \sum_{m=1}^{\infty} |b_m|^2 = \min_m q(m\pi)(u, u). \tag{8}$$

Since $a_n > 0$, $q(x)$ must have a greatest lower bound so that the conclusion of the lemma follows from (8).

N.B. (8) yields a lower bound for the smallest eigenvalue of L_0.

PROOF OF THE THEOREM. We assume that 0 is not an eigenvalue for L_1 or L_0 and that in fact L_0 is a positive definite operator. Otherwise we can shift L_1 and L_0 by a constant to accomplish this. Now we can associate a Green's function with both operators. Denote these by $K_1(x, y)$ and $K_0(x, y)$ respectively. These satisfy the equations

$$L_0 K_0(x, y) = \delta(x - y), \tag{9}$$

$$L_1 K_1(x, y) = \delta(x - y). \tag{10}$$

But it is evident since both have the same singular behavior at $x = y$, that

$$l(K_1(x, y) - K_0(x, y)) = 0. \tag{11}$$

(11) is in fact an ordinary differential equation so that

$$K_1(x, y) - K_0(x, y) = \sum_{j=1}^{2n} c_j(y) u_j(x) \tag{12}$$

where the $u_j(x)$ are $2n$ linearly independent solutions of $lu=0$, and the $c_j(y)$ suitable functions of y. Since $K_0(x, y)$ and $K_1(x, y)$ are symmetric in x and y each $c_j(y)$ is expressible in terms of the $u_j(y)$ so that (12) can be rewritten as

$$K_1(x, y) - K_0(x, y) = \sum_{i,j=1}^{2n} \gamma_{ij} u_i(y) u_j(x) = D(x, y), \qquad \gamma_{ij} = \gamma_{ji}. \tag{13}$$

Let K_1, K_0, D denote integral operators associated with the above kernels. Then

$$(K_1 u, u) = (K_0 u, u) + (Du, u) \geqq (Du, u) \tag{14}$$

since K_0 is positive definite. D is a degenerate operator and therefore semibounded so that (14) proves the theorem.

N.B. (14) shows that K_1 has at most $2n$ negative eigenvalues.

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