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BOUNDARY ZERO SETS OF A^∞ FUNCTIONS SATISFYING GROWTH CONDITIONS

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ABSTRACT. Let A denote the algebra of functions analytic in the open unit disc D and continuous in $\bar{D}$, and let

$$A^\infty = \{f \in A : f^{(n)} \in A, n = 0, 1, 2, \cdots\}.$$

For $f \in A$ denote the set of zeros of f in $\bar{D}$ by $Z^0(f)$, and for $f \in A^\infty$ let $Z^\infty(f) = \bigcap_{n=0}^\infty Z^0(f^{(n)})$. We study the boundary zero sets of A^∞ functions F satisfying, for some sequence $\{M_n\}$ and some $B>0$,

$$(1) \qquad |F^{(n)}(z)| \leqq n!\, B^n M_n, \qquad z \in \bar{D},\ n = 0, 1, 2, \cdots.$$

In particular, when $M_n = \exp(n^p)$, $p>1$, it is shown that for E, a proper closed subset of ∂D, there exists $F \in A^\infty$ satisfying (1) and with $Z^0(F) = Z^\infty(F) = E$ if and only if $\int_{-\pi}^{\pi} |\log \rho(e^{i\theta}, E)|^q \, d\theta < +\infty$. Here $\rho(z, E)$ is the distance from z to E and $(1/p)+(1/q)=1$.

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Before outlining the construction which yields the result stated above, let us recall some known facts and make a few simple observations. If $f \in A$, $f \not\equiv 0$, and satisfies a Lipschitz condition of order α, $|f(z)-f(z')| \leqq$

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$C|z-z'|^\alpha$, then $\log|f(z)|\leqq\alpha\log\rho(z, Z^0(f))+\log C$; and, consequently, $\int_{-\pi}^{\pi}\log\rho(e^{i\theta}, Z^0(f))\,d\theta>-\infty$ by Riesz's theorem. Conversely, Carleson [**1**] showed that if $E\subset\partial D$ is closed and

$$\int_{-\pi}^{\pi}\log\rho(e^{i\theta}, E)\,d\theta > -\infty, \tag{2}$$

then for any $m>0$ there exists an outer function

$$F\in A^m = \{f\in A : f, f', \cdots, f^{(m)}\in A\}$$

such that $Z^0(F)=Z^0(F')=\cdots=Z^0(F^{(m)})=E$. This result has been extended to show that there exists an F in A^∞ with $Z^0(F)=Z^\infty(F)=E$ (see [**5**], [**6**], or [**7**]). The extension is also a consequence of a recent theorem of Carleson and S. Jacob, which implies that an outer function $F\in A$ with $|F|\in C^\infty(\partial D)$ belongs to A^∞.

In case F satisfies the stronger hypothesis (1) we can say more. For, if $F\in A^\infty$, $F\not\equiv 0$, and $E=Z^\infty(F)$, then it follows from Taylor's formula with remainder that

$$|F(z)| \leqq (n!)^{-1}\rho(z, E)^n \max\{|F^{(n)}(z)| : z\in D\}, \qquad n=0, 1, 2, \cdots.$$

Thus, because of (1), $|F(z)|\leqq\rho(z, E)^n B^n M_n$, so that

$$-\log|F(e^{i\theta})| \geqq \sup\{-n\log\rho(e^{i\theta}, E) - \log B^n M_n : n=0, 1, 2, \cdots\}.$$

The integrability of $\log|F(e^{i\theta})|$ then implies that

$$\int_{-\pi}^{\pi} g^*(-\log\rho(e^{i\theta}, E))\,d\theta < +\infty \tag{3}$$

where $g^*(x)=\sup\{nx-\log B^n M_n : n=0, 1, 2, \cdots\}$. This was already noted by Carleson [**1**, p. 330] (with similar proof) in case $M_n=(n!)^\alpha$. See also A. Chollet [**2**].

It is not to be expected that (3) is, in general, a sufficient condition for the existence of $F\in A^\infty$ satisfying (1) and with $Z^0(F)=Z^\infty(F)=E$. For example, in the case $E=\{1\}$, it is known [**4**, Theorem 1, equation 6] that the necessary and sufficient condition is

$$\int_{-\pi}^{\pi} h^*(-2\log\rho(e^{i\theta}, E))\,d\theta < +\infty \tag{4}$$

where $h^*(x)=\sup\{nx-\log n!\,B^n M_n : n=0, 1, \cdots\}$. In particular, if $M_n=n!\,(\log(n+1))^{kn}$ with $1<k\leqq 2$, then the integral (4) diverges while the integral (3) converges.

Our construction of A^∞ outer functions satisfying a growth condition of form (1) is based on the following theorem. As above, E is a proper closed subset of ∂D and $\rho(z)=\rho(z, E)$ is the distance from z to E. Also, if $\{(e^{ia_n}, e^{ib_n})\}$ are the complementary arcs of E in ∂D, define

$$\tilde{\rho}(\theta) = \frac{1}{2\pi}\left(\frac{1}{\theta - a_n} + \frac{1}{b_n - \theta}\right)^{-1}, \qquad \theta \in (a_n, b_n)$$
$$= 0, \qquad e^{i\theta} \in E.$$

Note that $(4\pi)^{-1}\rho(e^{i\theta}) \leqq \tilde{\rho}(\theta) \leqq \frac{1}{4}\rho(e^{i\theta}) \leqq \frac{1}{2}$.

THEOREM 1. *Let λ^* be a nonnegative convex infinitely differentiable function such that $\varphi(e^{i\theta})=\lambda^*(-2\log\tilde{\rho}(\theta))$ satisfies*

(i) $(1/2\pi)\int_{-\pi}^{\pi}|\varphi(e^{i\theta})|\,d\theta \leqq M < +\infty$ *for some constant M*;

(ii) $|(d^n/d\theta^n)\varphi(e^{i\theta})| \leqq n!\,K^{n+1}\rho(e^{i\theta})^{-n-1}$, $e^{i\theta} \in \partial D \sim E$, $n=0, 1, 2, \cdots,$ *for some constant $K>0$*;

(iii) *for every constant $C>0$, $\varphi(e^{i\theta})+C\log\rho(e^{i\theta}) \to +\infty$ as $\rho(e^{i\theta}) \to 0$.*

Then there exists an outer function $F \in A^\infty$ with $Z^0(F)=Z^\infty(F)=E$ and a constant $B>0$ such that

$$|F^{(n)}(z)| \leqq n!\,B^n e^{\lambda(n)}, \qquad n = 0, 1, \cdots, \tag{5}$$

where $\lambda(n)=\sup\{nx-\lambda^*(x) : x>0\}$.

PROOF. Let

$$G(z) = G(z, \varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,\varphi(e^{i\theta})\,d\theta, \qquad z \in D,$$

and let $F=\exp(-G)$. We first assert that the derivatives of G satisfy, for some $K_0 \geqq 1$, $|G^{(n)}(z)| \leqq n!\,K_0^{n+1}\rho(z)^{-n-1}$, $n=0, 1, 2, \cdots$. This may be proved by repeating the proof of Lemma 2.3 of **[6]** and keeping track of the constants which appear there. We omit the details of this computation. In particular, we have the slightly weaker estimate

$$|G^{(n+1)}(z)| \leqq n!\,(2K_0^2)^{n+1}\rho(z)^{-n-2}, \qquad n=0, 1, 2, \cdots.$$

Next we claim that

$$|F^{(n)}(z)| \leqq n!\,(4K_0^2)^{n+1}\,|F(z)|\,\rho(z)^{-2n}, \qquad n = 0, 1, \cdots. \tag{6}$$

The proof is by induction on n. Now (6) is clear for $n=0$. Assume (6) for $n=0, \cdots, j$. For $n=j+1$,

$$|F^{(j+1)}(z)| = \left|\frac{d^j}{dz^j}F(z)G'(z)\right| \leqq \sum_{n=0}^{j}\binom{j}{n}|F^{(j-n)}(z)G^{(n+1)}(z)|$$
$$\leqq j!\,2^{j+3}(K_0^2)^{j+2}\,|F(z)|\sum_{n=0}^{j}2^{j-n}\rho(z)^{-2(j+1)+n}.$$

Since $\rho(z)\leqq 2$, $2^{j-n}\rho(z)^{-2(j+1)+n}\leqq 2^{j+1}\rho(z)^{-2(j+1)}$. Hence

$$\sum_{n=0}^{j} 2^{j-n}\rho(z)^{-2(j+1)+n} \leqq (j+1)2^{j+1}\rho(z)^{-2(j+1)},$$

and (6) follows.

Because $|F^{(n)}(z)|\leqq D_n\rho(z)^{-2n}$ for some constant $D_n>1$,

$$\log |F^{(n)}(re^{i\theta})| \leqq -2n \log \rho(re^{i\theta}) + \log D_n,$$

and so

$$\begin{aligned}\log^+ |F^{(n)}(re^{i\theta})| &\leqq -2n \log \rho(re^{i\theta}) + \log D_n + 2n \log 2\\ &\leqq -2n \log \rho(e^{i\theta}) + \log D_n + 4n \log 2,\end{aligned}$$

where the last inequality follows from $\rho(e^{i\theta})\leqq 2\rho(re^{i\theta})$.

Since $\log \rho(e^{i\theta})$ is integrable, $F^{(n)}$ is of bounded characteristic on D (i.e. of class N). Moreover, the dominated convergence theorem implies that

$$\lim_{r\to 1^-} \int_{-\pi}^{\pi} \log^+ |F^{(n)}(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} \log^+ |F^{(n)}(e^{i\theta})|\, d\theta.$$

Consequently, $F^{(n)}$ has the factorization $B_nS_nH_n$ where B_n is a Blaschke product, S_n is a singular inner function, and H_n is an outer function for the class N. See e.g. [3, p. 26]. Thus $F^{(n)}$ has the bound (5) iff the boundary values of $F^{(n)}$ have this bound. By (6),

$$|F^{(n)}(e^{i\theta})| \leqq n!\,(4K_0^2)^{n+1}\, |F(e^{i\theta})|\, \rho(e^{i\theta})^{-2n} \quad \text{a.e.}$$

Hence, for some constant $B>0$,

$$|F^{(n)}(e^{i\theta})| \leqq n!\, B^n\, |F(e^{i\theta})|\, \tilde{\rho}(\theta)^{-2n} \quad \text{a.e.}$$

or

$$\begin{aligned}|F^{(n)}(e^{i\theta})| &\leqq n!\, B^n \exp[-2n \log \tilde{\rho}(\theta) - \lambda^*(-2 \log \tilde{\rho}(\theta))] \quad \text{a.e.}\\ &\leqq n!\, B^n e^{\lambda(n)}.\end{aligned} \tag{7}$$

This establishes (5) and also shows that $F\in A^\infty$. It is clear from the definition of F, (iii), and (7) that $Z^0(F)=Z^\infty(F)=E$.

THEOREM 2. *Let E be a proper closed subset of ∂D. A necessary and sufficient condition that there exists $F\in A^\infty$ with $Z^0(F)=Z^\infty(F)=E$ and a constant $B>0$ such that*

$$|F^{(n)}(z)| \leqq n!\, B^n e^{n^p}, \qquad n = 0, 1, \cdots, \tag{8}$$

where $p>1$, is that $\int_{-\pi}^{\pi} |\log \rho(e^{i\theta}, E)|^q\, d\theta<+\infty$, $(1/p)+(1/q)=1$.

PROOF. Assuming the existence of such an F, (3) holds with $g^*(x)= \sup\{nx-n \log B-n^p : n=0, 1, \cdots\}$. A routine calculation shows $X^2= O(g^*(x))$ for large x. Hence $|\log \rho(e^{i\theta})|^q$ is integrable.

For the converse we apply Theorem 1 with $\lambda^*(x)=(p/q)(x/p)^q$. For this λ^*, straightforward calculations verify that the hypotheses of Theorem 1 are satisfied and that $\lambda(n)=n^p$.

Theorem 1 also gives information in some cases when we do not know that (3) is a sufficient condition. For example, the following theorem, due to A. Chollet [2], may be obtained.

THEOREM 3. *Let E be a proper closed subset of ∂D. If there exists $F\in A^\infty$, $F\not\equiv 0$, with $Z^\infty(F)\supset E$ and a constant $B>0$ such that*

$$|F^{(n)}(z)| \leqq B^n(n!)^\alpha, \qquad n=0,1,\cdots, \tag{9}$$

where $\alpha>1$, then

$$\int_{-\pi}^{\pi} \rho(e^{i\theta},E)^{-1/(\alpha-1)}\,d\theta < +\infty. \tag{10}$$

In the converse direction, if $\alpha>2$ and (10) holds, then there exists $F\in A^\infty$ with $Z^0(F)=Z^\infty(F)=E$ and a constant $B>0$ such that $|F^{(n)}(z)|\leqq B^n(n!)^{2\alpha-1}$, $n=0,1,\cdots$.

PROOF. If $F\in A^\infty$ with $Z^\infty(F)\supset E$ satisfies (9), then (3) holds with $g^*(x)=\sup\{nx-\log B^n(n!)^{\alpha-1}:n=0,1,\cdots\}$. Since $e^{x/(\alpha-1)}=O(g^*(x))$ for large x, (3) implies (10). In the converse direction apply Theorem 1 with $\lambda^*(x)=2e^{-1}(\alpha-1)e^{x/2(\alpha-1)}$. Then $\varphi(e^{i\theta})=2e^{-1}(\alpha-1)\tilde{\rho}(\theta)^{-1/(\alpha-1)}$ and is easily seen to satisfy (i), (ii), and (iii) of Theorem 1. A simple calculation shows

$$e^{\lambda(n)} = O(e^{2(\alpha-1)n}(n!)^{2(\alpha-1)}).$$

REMARK. Theorem 3 gives another proof that the class of A^∞ functions satisfying (9) for $1<\alpha<2$ is quasi-analytic.

REMARK. Mme. Chollet has sharpened the last part of Theorem 3 (unpublished) by showing that the exponent $2\alpha-1$ may be replaced by $2\alpha-2$.

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