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ORDINAL SUM-SETS

MARTIN M. ZUCKERMAN[1]

ABSTRACT. A finite set, B, of ordinals will be called a sum-set if there are nonzero ordinals $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that the set of sums of $\alpha_1, \alpha_2, \cdots, \alpha_n$, in all $n!$ permutations of the summands, is B. Let B_k denote an arbitrary k-element sum-set; we consider various matters related to the set of numbers n for which there are n summands for B_k.

1. In general, addition of ordinals depends on the order of the summands. Various studies, [1], [3], [4], [5], [6], [7], and [8], have been concerned with determining information about the sets E_n of natural numbers k for which there exist n (not necessarily distinct) ordinals that in all possible orderings yield k distinct sums. We now investigate the following related problem: We say that a set of k distinct ordinals $\{\beta_1, \beta_2, \cdots, \beta_k\}$ is a *sum-set* provided that there are a finite number of (not necessarily distinct) nonzero ordinals $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that the set of sums of $\alpha_1, \alpha_2, \cdots, \alpha_n$ (in all $n!$ arrangements) is $\{\beta_1, \beta_2, \cdots, \beta_k\}$. In this case we say that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are *summands* for $\{\beta_1, \beta_2, \cdots, \beta_k\}$. Thus, here we are concerned with which finite sets of ordinals are sets of sums of a finite number of ordinals.

2. Every nonzero ordinal α has a unique *Cantor normal form*:

$$\alpha = \omega^{\alpha_1}M_1 + \omega^{\alpha_2}M_2 + \cdots + \omega^{\alpha_r}M_r, \tag{1}$$

where $r, M_1, M_2, \cdots, M_r$ are nonzero natural numbers and where $\alpha_1 > \alpha_2 > \cdots > \alpha_r \geqq 0$. α_1 is called the *degree of* α, written "deg(α)", and M_1 is called the *leading coefficient of* α.

Every nonzero ordinal α can be uniquely represented in the form

$$\alpha = \omega^{\alpha_1}M_1 + \rho, \tag{2}$$

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in which, with reference to $r, M_1, M_2, \cdots, M_r, \alpha_1, \alpha_2, \cdots, \alpha_r$ in (1),

$$\begin{aligned} \rho &= 0 && \text{if } r = 1; \\ &= \omega^{\alpha_2} M_2 + \cdots + \omega^{\alpha_r} M_r && \text{otherwise.} \end{aligned}$$

We shall refer to (2) as the *remainder form representation of* α; ρ will be called the *remainder of* α, written "rem(α)". Clearly, if $\rho \neq 0$, then $\deg(\rho) < \alpha_1$. For any set A of ordinals, let $R_A = \{\text{rem}(\alpha) : \alpha \in A\}$.

A set A is said to be *compatible* if A is a set $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of nonzero ordinals and if each α_i, $1 \leqq i \leqq k$, has the same degree and leading coefficient. If A $(=\{\alpha_1, \alpha_2, \cdots, \alpha_k\})$ is compatible and if the leading coefficient of each α_i is M, we say that A is *M-compatible* and that $\deg(\alpha_1)$ is the *degree of* A, written "$\deg(A)$".

Throughout this section B will always denote a k-element set of ordinals $\{\beta_1, \beta_2, \cdots, \beta_k\}$, $k = 1, 2, \cdots$; moreover, $\beta_i = \omega^{\beta_{i,1}} M_i + \rho_i$ will be the remainder form representation of β_i, $i = 1, 2, \cdots, k$.

Clearly, every unit set B is a sum-set. If $B = \{\beta_1, \beta_2\}$ is compatible, we can suppose $\beta_1 < \beta_2$. Then β_1 and $\rho_2 - \rho_1$ are summands for B.

For any set A, let $\mathscr{P}^*(A)$ be the set of nonempty subsets of A. If $J \in \mathscr{P}^*(\{1, 2, \cdots, n\})$, let Σ_J be the group of permutations of J. In particular, if $J = \{1, 2, \cdots, n\}$, we write "Σ_n" instead of "Σ_J".

Let R be a set of ordinals. We shall say that C is an *additive derivative of* R if there exist ordinals $\gamma_1, \gamma_2, \cdots, \gamma_m$ for which

$$(3) \quad C = \left\{ \rho + \sum_{j \in J} \gamma_{\phi(j)} : \rho \in R,\ J \in \mathscr{P}^*(\{1, 2, \cdots, m\}), \text{ and } \phi \in \Sigma_J \right\}.$$

THEOREM 1. *B is a sum-set if and only if for some l, M satisfying $1 \leqq l \leqq M < \infty$,*

(i) *B is M-compatible, and*

(ii) *R_B is an additive derivative of R_{B_0} for some l-element subset B_0 of B.*

PROOF. Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are summands for B. Let the remainder form representation of α_i be $\alpha_i = \omega^{\deg(\alpha_i)} M_i + r_i$ for $i = 1, 2, \cdots, n$. Let $\alpha^* = \max\{\deg(\alpha_i) : 1 \leqq i \leqq n\}$, let $A = \{i : 1 \leqq i \leqq n$ and $\deg(\alpha_i) = \alpha^*\}$, and let $M = \sum_{i \in A} M_i$. Clearly, A has M or fewer elements. For each $\phi \in \Sigma_n$, let j_ϕ be the largest number i for which $\phi(i) \in A$. Then

$$(4) \qquad \sum_{i=1}^{n} \alpha_{\phi(i)} = \omega^{\alpha^*} M + r_{j_\phi} + \alpha',$$

where

$$\begin{aligned} \alpha' &= 0 && \text{if } j_\phi = n, \\ &= \sum_{i = j_\phi + 1}^{n} \alpha_{\phi(i)} && \text{if } j_\phi < n. \end{aligned}$$

In the latter case, $\deg(\alpha')<\alpha^*$. Therefore, $B=\{\sum_{i=1}^{n} \alpha_{\phi(i)}:\phi \in \Sigma_n\}$ is M-compatible.

Let $B_0=(\omega^{\alpha^*}M+r_i . i \in A\}$; clearly, B_0 is an l-element subset of B for $1\leqq l\leqq M$. Let $\gamma_1, \gamma_2, \cdots, \gamma_{m-1}$ be those ordinals among $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which $i \notin A$ (if any such exist); let $\gamma_m=0$. According to (3) and (4), R_B is an additive derivative of R_{B_0}.

Now suppose that for $1\leqq l\leqq M<\infty$, conditions (i) and (ii) hold; we can further suppose that $B_0=\{\beta_{u_1}, \beta_{u_2}, \cdots, \beta_{u_l}\}$ and that $\gamma_1\leqq\gamma_2\leqq\cdots\leqq\gamma_m$ are ordinals for which

$$R_B = \left\{\rho_{u_i} + \sum_{j\in J} \gamma_{\phi(j)}:1 \leqq i \leqq l, J \in \mathscr{P}^*(\{1, 2, \cdots, m\}) \text{ and } \phi \in\Sigma_J\right\}.$$

Surely, $\gamma_1=0$. Moreover, for each $J\in \mathscr{P}^*(\{1, 2, \cdots, m\})\sim\{1\}$ and for each $\phi \in \Sigma_J$, $\deg(\sum_{j\in J} \gamma_{\phi(j)})<\deg(B)$. For $i=1, 2, \cdots, l+m-1$, let

$$\begin{aligned} \alpha_i &= \omega^{\deg(B)} + \rho_{u_i} && \text{if } 1 \leqq i < l; \\ &= \omega^{\deg(B)}(M - l + 1) + \rho_{u_l} && \text{if } i = l; \\ &= \gamma_{i-l+1} && \text{if } l < i < l + m. \end{aligned}$$

Then $\alpha_1, \alpha_2, \cdots, \alpha_{l+m-1}$ are summands for B.

THEOREM 2. *Every compatible set B can be extended to a sum-set.*

PROOF. Let B be M-compatible. Consider the set B^* for which $\omega^{\deg(B)}M, \rho_1, \rho_2, \cdots, \rho_k$ are the summands. Surely, $B\subseteq B^*$.

3. For each nonzero natural number k, let B_k denote an arbitrary k-element sum-set, and let $N(B_k)$ be the set of natural numbers n for which there are n (not necessarily distinct) summands for B_k. For positive natural numbers k and M, let $N_{\text{int}}(k, M)=\bigcap \{N(B_k)$:all M-compatible $B_k\}$; let $N_{\max}(k, M)=\max\{N(B_k)$:all M-compatible $B_k\}$ (if this maximum exists); let $N_{\min}(k, M)=\min\{N(B_k)$:all M-compatible $B_k\}$.

THEOREM 3. *For all nonzero natural numbers k and M*, $N_{\max}(k, M)$ *exists and*

$$N_{\max}(k, M) = k + M - 1.$$

PROOF. For any k, M, consider $\beta_i=\omega M+(i-1)$, $i=1, 2, \cdots, k$. Let $\alpha_i=\omega$ for $1\leqq i\leqq M$ and, if $k>1$, let $\alpha_i=1$ for $M+1\leqq i\leqq M+k-1$. Then $\alpha_1, \alpha_2, \cdots, \alpha_{M+k-1}$ are summands for $\{\beta_1, \beta_2, \cdots, \beta_k\}$. Thus $N_{\max}(k, M)\geqq k+M-1$.

Let $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ be an arbitrary k-element M-compatible set. Let the remainder form representation of γ_i be $\gamma_i=\omega^\gamma M+\rho_i$, $i=1, 2, \cdots, k$, where $\rho_1<\rho_2<\cdots<\rho_k$. Suppose $\delta_1, \delta_2, \cdots, \delta_L$ are summands for $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$. At least one and at most M of these δ_i are of degree γ. For some j, $1 \leqq j \leqq L$, suppose $\deg(\delta_j)=\gamma$; let $\rho_j=\text{rem}(\delta_j)$. For any nonzero natural number r, let $0<\sigma_1<\sigma_2<\cdots<\sigma_r<\omega^\gamma$. Then the ordinals $\omega^\gamma M+\rho_j$, $\omega^\gamma M+\rho_j+\sigma_1$, $\omega^\gamma M+\rho_j+\sigma_1+\sigma_2, \cdots, \omega^\gamma M+\rho_j+\sigma_1+\sigma_2+\cdots+\sigma_r$ are all distinct. Thus there are at most $k-1$ summands for $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ of degree less than γ. Consequently, $L \leqq M+k-1$, and $N_{\max}(k, M)=k+M-1$.

THEOREM 4. *Let* $M \geqq 1$.

(a) *For all* B_1, $N(B_1)=\{1, 2, \cdots, M\}$;

(b) *for all* B_2, $N(B_2)=\{2, 3, \cdots, M+1\}$;

(c) *for* $3 \leqq k \leqq M$, $N_{\text{int}}(k, M)=N(\{\omega M+2^{i-1}: i=1, 2, \cdots, k\})=\{k, k+1, \cdots, M\}$.

PROOF. For $k=1, 2, \cdots$, assume $M \geqq k$; let $B_k=\{\beta_1, \beta_2, \cdots, \beta_k\}$, where $\beta_i=\omega^\beta M+\rho_i$ is the remainder form representation of β_i, $i=1, 2, \cdots, k$. For $k \leqq j \leqq M$, let $\alpha_i=\omega^\beta+\rho_i$ for $1 \leqq i<j$ and let $\alpha_j=\omega^\beta(M-j+1)+\rho_j$. Clearly, for each such j, $\alpha_1, \alpha_2, \cdots, \alpha_j$ are summands for B_k; hence $\{k, k+1, \cdots, M\} \subseteq N(B_k)$ for each B_k.

(a) and (b). For $k=2$, assume $\beta_1<\beta_2$. For $M=1, 2, \cdots$, let $\gamma_1=\gamma_2=\cdots=\gamma_M=\omega^\beta+\rho_1$ and let $\gamma_{M+1}=\rho_2-\rho_1$; then $\gamma_1, \gamma_2, \cdots, \gamma_{M+1}$ are summands for B_2. Theorem 3 now guarantees that $N(B_1)=\{1, 2, \cdots, M\}$ and $N(B_2)=\{2, 3, \cdots, M+1\}$.

(c). In order to show that for $3 \leqq k \leqq M$, $N_{\text{int}}(k, M)=\{k, k+1, \cdots, M\}$, it suffices to show that $N(\{\omega M+2^{i-1}: i=1, 2, \cdots, k\}) \subseteq \{k, k+1, \cdots, M\}$—hence $=\{k, k+1, \cdots, M\}$, by the above. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be summands for $\{\omega M+2^{i-1}: i=1, 2, \cdots, k\}$.

Suppose that at least one of the α_i—say α_n—is finite. Clearly, at least one of the α_i must be infinite.

Case 1. There are at least two distinct remainders among the infinite α_i. Then one of these must be 1; another is of the form 2^l for $l \geqq 1$. Then $1+\alpha_n=2^m$ for $m \geqq 1$; therefore α_n is odd. Also, $2^l+\alpha_n=2^p$ for some $p \geqq 1$; therefore α_n is even. Contradiction!

Case 2. All of the infinite α_i have the same remainder, ρ. Then ρ must be 1. Let j be the number of infinite α_i; we can suppose $\alpha_{j+1} \leqq \alpha_{j+2} \leqq \cdots \leqq \alpha_n<\omega$. We must have $\alpha_{j+1}=1$ in order to yield the sum $\omega M+2$. Similarly, we must have $j+2 \leqq n$ and α_{j+2} equal to 2 or 3—in order to yield the sum $\omega M+4$. But either of these choices for α_{j+2} would yield a sum—$\omega M+3$ or $\omega M+5$—that is not in $\{\omega M+2^{i-1}: i=1, 2, \cdots, k\}$.

Thus all of the α_i must be infinite. If $n>M$, then since each α_i is of degree 1, the leading coefficient of any sum is at least n—hence is bigger than M. If $n<k$, then there are fewer than k remainders obtained among the $n!$ permutations.

COROLLARY. *If* $\gamma_i = \omega M + 2^{i-1}$, $i = 1, 2, \cdots, k$, *and* $k \geqq \max(3, M+1)$, *then* $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ *is not a sum-set.*

4. The determination of $N_{\min}(k, M)$ is considerably more difficult! It is closely related to problems encountered in [7].

THEOREM 5. *Let* k, M, *and* m *be arbitrary nonzero natural numbers. Then* $N_{\min}(k, M+m) \leqq N_{\min}(k, M)$.

PROOF. Let $\beta_i = \omega^\beta M + \rho_i$ be the remainder form representation of β_i, $i = 1, 2, \cdots, k$. Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are summands for the k-element set $\{\beta_1, \beta_2, \cdots, \beta_k\}$. At least one of the ordinals α_i, $1 \leqq i \leqq n$, is of degree β; let α_{i_0} be any such ordinal and let the remainder form representation of α_{i_0} be $\alpha_{i_0} = \omega^\beta M_{i_0} + r_{i_0}$. Let $\alpha^*_{i_0} = \omega^\beta(M_{i_0} + m) + r_{i_0}$, and for $i \neq i_0$ and $1 \leqq i \leqq n$, let $\alpha^*_i = \alpha_i$. Let $\beta^*_i = \omega^\beta(M+m) + \rho_i$, $i = 1, 2, \cdots, k$. Then $\alpha^*_1, \alpha^*_2, \cdots, \alpha^*_n$ are summands for the k-element set $\{\beta^*_1, \beta^*_2, \cdots, \beta^*_k\}$.

For each nonzero natural number k we let

$$N_{\min}(k) = \min\{N_{\min}(k, M) : M = 1, 2, \cdots\}.$$

We note that $N_{\min}(k) = N_{\min}(k, 1)$ for $1 \leqq k \leqq 449$. (See [3], [4], [5, pp. 265–266], and [7, proofs of Theorems 2, 3, and 4].) Theorems 3, 5, and 6 together imply that for all natural numbers k and M, except for $k=1$, $M=1$ and for $k=2$, $M=1$,

$$N_{\min}(k, M) < N_{\max}(k, M).$$

THEOREM 6. *Let* k *and* n *be nonzero natural numbers. Let*

$$L(n) = \max\{l : (l+2)^2 2^l - 4(l+2) \leqq (l+1)2^{n-l}\}.$$

If $k \leqq (L(n)+2)2^{n-2} - 2^{L(n)} + 1$, *then* $N_{\min}(k) \leqq N_{\min}(k, 1) \leqq n$.

PROOF. This follows from [7, Theorem 2].

For example, if $n=6$, then $L(n)=2$; Theorem 6 indicates that for $k \leqq 61$, $N_{\min}(k) \leqq 6$.

For each nonzero natural number n, the maximum number m_n of distinct values that can be assumed by a sum of n nonzero ordinals in all $n!$ permutations of the summands has been calculated in [1] and [5]. The numbers m_n increase with n; for $n \geqq 3$, $m_n < n!$. It is easily seen from the formulas given by Erdös and Wakulicz that for $n \geqq 10$, $n \neq 14$,

$$m_n = 3^{4(k-(l \dot{-} 1)) - 3(1 \dot{-} l)} 11^{1 \dot{-} l} 193^{l \dot{-} 1},$$

where $n = 5k + l$ for k, l nonnegative integers with $l \leqq 4$, and where for nonnegative integers r and s,

$$r \dot{-} s = r - s, \qquad r \geqq s,$$
$$= 0, \qquad r < s.$$

It immediately follows that if $k>m_n$, then $N_{\min}(k)>n$. Moreover, [3], [4], [5], [6], and [7, Theorem 4] yield that for $1\leqq k\leqq 29$ and for $31\leqq k\leqq 449$, if n is such that $m_{n-1}<k\leqq m_n$, then $N_{\min}(k)=n$.

We note that $N_{\min}(29)=N_{\min}(31)=5$, whereas $N_{\min}(30)=6$; thus $N_{\min}$ is not monotonic. Furthermore, [8, Theorem 1] indicates that there are infinitely many k for which there are j and l satisfying $j<k<l$ and $N_{\min}(j)=N_{\min}(l)<N_{\min}(k)$.

Let $n\geqq 1$. Let

$$\mathscr{B}_n = \{\{\langle l_1, r_1, 1\rangle, \langle l_2, r_2, 2\rangle, \cdots, \langle l_n, r_n, n\rangle\}: r_1 = 0 \text{ and for } i = 1, 2, \cdots, n-1, l_i \geqq 1, \text{ and } r_{i+1} = r_i \text{ or } r_{i+1} = r_i + 1\}.$$

For each $B\in\mathscr{B}_n$ and for each $j=0, 1, \cdots, n-1$, let $B_j=\{\langle l_i, i\rangle:\langle l_i, j, i\rangle\in B\}$ and let $B_j^{\#}$ be the number of distinct sums $\sum l_i$ such that $\langle l_i, i\rangle\in B_j'\cup B_j''$, where B_j' ranges over nonempty subsets of B_j and B_j'' ranges over subsets (possibly empty) of $\bigcup\{B_u:u\neq j \text{ and } 0\leqq u\leqq n-1\}$. Let $B^{\#}=\sum_{j=0}^{n-1} B_j^{\#}$, let $\mathscr{C}_n=\bigcup\{B^{\#}:B\in\mathscr{B}_n\}$, and let $\mathscr{D}_n=\{1+x:x\in\mathscr{C}_n\}$.

THEOREM 7. *Let k and n be nonzero natural numbers. If there exist nonzero natural numbers $n_1, n_2, \cdots, n_s$; $k_1, k_2, \cdots, k_s$ $(s\geqq 1)$ such that $n=\sum_{i=1}^{s} n_i$ and $k=\prod_{i=1}^{s} k_i$, where $k_1\leqq n_1$ and for $i=2, 3, \cdots, s$, $k_i\in\mathscr{D}_{n_i}$, then $N_{\min}(k, n_1)\leqq n$.*

PROOF. See the proof of [7, Theorem 1].

THEOREM 8. *Let k, m, and M be nonzero natural numbers.*

(a) *Let $s\in\mathscr{D}_m$. Then*

$$N_{\min}(sk, M) \leqq m + N_{\min}(k, M).$$

In particular,

$$N_{\min}(2k, M) \leqq 1 + N_{\min}(k, M).$$

(b) *Let $1\leqq s\leqq m$. Then for some M', $M<M'\leqq 2M$,*

$$N_{\min}(sk, M') \leqq m + N_{\min}(k, M).$$

PROOF. This follows from [7, Theorem 3].

THEOREM 9. *Let k and M be nonzero natural numbers. Then for some M', $M<M'\leqq 2M$,*

$$N_{\min}(k + 1, M') \leqq 1 + N_{\min}(k, M).$$

PROOF. This follows from [7, Theorem 4].

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DEPARTMENT OF MATHEMATICS, CITY COLLEGE OF THE CITY UNIVERSITY OF NEW YORK, NEW YORK, NEW YORK 10031