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QUASI-COTRIPLEABLE CATEGORIES

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ABSTRACT. A category is quasi-cotripleable over the category of sets if it has all the properties of cotripleable categories except the right adjoint to the forgetful functor. Problems involving such categories are illustrated by categories of relational structures, and by categories of sets acted on by a monoid with open homomorphisms for maps. A characterization is given in terms of generalized operators and relations.

1. **Introduction.** In universal algebra, the natural object of study is a category tripleable over the category $\mathscr{S}$ of sets. One would therefore expect universal coalgebra to be the study of cotripleable categories. However, many natural constructions produce categories which are "almost" cotripleable, except that the forgetful functor lacks a right adjoint. The intent of this paper is thus to suggest the following class of categories as an object of study.

DEFINITION. A category $\mathscr{A}$ equipped with a faithful functor $U:\mathscr{A}\to\mathscr{S}$ is *quasi-cotripleable* (QCT) if

(1) $\mathscr{A}$ has coequalizers and (infinite) coproducts;

(2) U preserves coequalizers and coproducts;

(3) the dual of Beck's precise tripleableness condition holds (see, for example, [3] for an exposition). We shall express this condition by saying that $\mathscr{A}$ "has split equalizers."

Clearly a QCT category is cotripleable iff U satisfies the cosolution-set condition. As a first example of a QCT category which is not cotripleable we may cite the category of topological spaces and continuous open maps, see [5].

2. **Relational structures.** Let R be a binary relation symbol. There is a one-to-one correspondence between relations R on a set A and functions $\omega:A\to 2^A=PA$, given by $R(x,y)$ iff $y\in\omega(x)$. Let $\mathscr{A}$ be the category of sets equipped with a binary relation, and homomorphisms $f:A\to B$ satisfying

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$\omega_B f(a)=f(\omega_A a)$ (direct image) for all a in A. In terms of R the condition on homomorphisms means: $R(a, a')$ implies $R(fa, fa')$, and $R(fa, b)$ implies $b=f(a')$ where $R(a, a')$. With the obvious forgetful functor, $\mathscr{A}$ is QCT.

THEOREM 1. *The category $\mathscr{A}$ is not cotripleable.*

PROOF. We show there is no cosolution-set for $X=\{0, 1\}$. For each ordinal α we define $P^\alpha X$ by $P^0X=X$, $P^{\alpha+1}X=P(P^\alpha X)$. If α is a limit ordinal we define $P^\alpha X$ to be the direct limit of $P^\beta X$, $\beta<\alpha$, under the system of maps determined inductively by $P^\beta X \to P^{\beta+1}X$ given by $x \mapsto \{x\}$. Let ∞ be an arbitrarily large limit ordinal and define A to be the subset of $\prod_{\alpha<\infty} P^\alpha X$ consisting of all sequences $x=(x_\alpha)_{\alpha<\infty}$ such that (1) for each α, $x_\alpha \in x_{\alpha+1}$; and (2) if α is a limit ordinal there is some $\beta<\alpha$ so that for $\beta<\gamma<\delta\leqq\alpha$ we have $x_\delta=p_\gamma^\delta(x_\gamma)$, where p_γ^δ is the evident map in the directed system. This set A can be arbitrarily large. Define $f: A\to X$ by $f(x)=x_0$, and define R on A by $R(x, y)$ iff for each α, $y_\alpha \in x_{\alpha+1}$.

Let $g: A\to B$ be a homomorphism such that $g(x)=g(y)$ implies $x_0=y_0$, and assume that g is not one-to-one. If $g(x)=g(y)$ and $x\neq y$, there is a least β such that $x_\beta\neq y_\beta$. We may assume that x and y are chosen so that this β is the least possible. Then $\beta\neq 0$, and by (2) β is not a limit ordinal, so $\beta-1$ exists. We may assume that $x_\beta \nsubseteq y_\beta$, and we can choose z in A such that $R(x, z)$ and $z_{\beta-1}\in x_\beta - y_\beta$. Since g is a homomorphism there must exist w in A such that $R(y, w)$ and $g(z)=g(w)$. But $z_{\beta-1}\neq w_{\beta-1}$ and this contradicts the minimality of β. $\square$

Many examples of QCT categories arise as subcategories of $\mathscr{A}$. Let P be a first order sentence in the predicates R and $=$, and let $\mathscr{A}(P)$ be the full subcategory of $\mathscr{A}$ whose objects are the models of P. We shall say that P is *admissible* if $\mathscr{A}(P)$ is QCT. It appears to be an interesting and difficult problem to characterize structurally the admissible sentences. Thus, $\forall x R(x, x)$ is admissible but the very similar sentence $\forall x \forall y R(x, y)$ is not admissible since it is not preserved under disjoint union.

If P is universal, then $\mathscr{A}(P)$ has split equalizers. Thus, as a first approach to the problem, we shall show that a fairly large class of universal sentences is admissible. However, an admissible sentence need not be universal, as is shown by the example $\forall x \exists ! y R(x, y)$. It is also easy to see that $\mathscr{A}(P)$ has coequalizers preserved by U if P is positive [2], but weaker conditions are sufficient. Some restriction is necessary, since coequalizers are not preserved if P is $\forall x \neg R(x, x)$, or if P is the sentence stating that R is an antisymmetric relation. Finally, P must be preserved by disjoint union. Thus, P could be of the form $A(x_1, \cdots, x_n)\Rightarrow B(x_1, \cdots, x_n)$ where all x_i actually occur in A and where the holding of A in a disjoint union implies that all x_i come from the same component. The latter will be true if A has the form $R(x_1, x_{11})\wedge R(x_1, x_{12})\wedge\cdots\wedge R(x_1, x_{1m})\wedge R(x_{11}, x_{111})\wedge\cdots$, where the x's are all distinct. Such a formula asserts that R arranges the x's into a tree diagram, so it will be called *dendritic*. The proof of the following result is now routine.

THEOREM 2. *A sentence P is admissible, provided it is the universal closure of an open formula of the form $A(x_1, \cdots, x_n)\Rightarrow B(x_1, \cdots, x_n)$ where:* (1) *A is dendritic*; (2) *B is positive*; (3) *every variable occurring in B also occurs in A.*

3. **Open homomorphisms.** Let M be a monoid with identity element e and A a left M-set. A subset of A will be called *open* if its complement is closed under the action of M, and an M-homomorphism $f: A\to B$ will be called open if it takes open sets to open sets (so in particular $f(A)$ must be open). Let $\mathcal{M}$ be the category of left M-sets and open homomorphisms. Then $\mathcal{M}$ is a QCT category; it appears to be difficult to determine when $\mathcal{M}$ is cotripleable. If M is a group then open sets are closed, every homomorphism is open, and $\mathcal{M}$ is cotripleable. We intend to show that $\mathcal{M}$ is cotripleable when M is finite cyclic. In obtaining the cosolution sets it clearly suffices to restrict attention to connected M-sets.

LEMMA 1. *Let M be commutative, A a connected M-set. If $a, b\in A$ then there exist m, n in M such that $ma=nb$.*

PROOF. Define a relation $\sim$ on A by $a\sim ma$ for a in A, m in M. Let $\equiv$ be the equivalence relation on A generated by $\sim$. Since A is connected, $a\equiv b$ for all a, b in A. If $b=ma$ we have $ma=eb$. Suppose by induction that $ma=nb$ and either $b\sim c$ or $c\sim b$. If $c\sim b$ then for some r in M, $c=rb$ and we have $ma=(nr)c$. If $b\sim c$ then $b=rc$ and we have $(rm)a=(rn)b=n(rb)=nc$. $\square$

Let M be finite cyclic; then $M=\{e, \sigma, \sigma^2, \cdots, \sigma^r, \cdots, \sigma^{n+r-1}\}$ where $\sigma^{n+r}=\sigma^r$. By [**1**, p. 20], $K=\{\sigma^r, \cdots, \sigma^{n+r-1}\}$ is a cyclic group of order n. Hence if A is an M-set then M acts like a group on the subset $\sigma^r A$.

LEMMA 2. *If A is a connected M-set and M is finite cyclic, then* $|\sigma^r A|\leqq|M|$.

PROOF. Let $a, b\in\sigma^r A$. By Lemma 1, we have $\sigma^s a=\sigma^t b$, and some element σ^m acts on $\sigma^r A$ as the inverse of σ^t. Hence $b=\sigma^m\sigma^t b=\sigma^{m+s}a\in Ma$, so we have $\sigma^r A=Ma$ and $|\sigma^r A|=|Ma|\leqq|M|$. $\square$

THEOREM 3. *If M is finite cyclic, the category of M-sets and open homomorphisms is cotripleable over the category of sets.*

PROOF. Let A be a connected M-set and $f: A\to X$. We begin an inductive process of identifying points of $A-(\sigma^r A)$ as follows. If $a, b\in A-(\sigma A)$, $\sigma a=\sigma b$, and $f(a)=f(b)$, then identify a with b. Next, suppose $a, b\in A-(\sigma^r A)$, $\sigma a=\sigma b$, $f(a)=f(b)$, and the process has already been applied to $M^{-1}s=\{x\in A|a\in Mx\}$ and to $M^{-1}b$, except for a and b themselves. Suppose there exists a one-to-one function k from $M^{-1}a$ onto $M^{-1}b$ such that $f(x)=f(k(x))$, and $k(\sigma x)=\sigma k(x)$ for $x\neq a$, and $k(a)=b$. Then we identify x with $k(x)$. Continue the process until $A-(\sigma^r A)$ is exhausted. At the conclusion we will have an identification map $g:A\to B$ where $|B|\leqq|M\times X\times P(X)\times\cdots\times P^r(X)|$. By the nature of the construction it is clear that g is an open homomorphism through which f factors. $\square$

EXAMPLE. Suppose M consists of e and an idempotent m. Define the M-set $A=\{a,b\}$ by $ma=mb=b$. Then A is "almost" a terminal object in that each M-set admits at most one open homomorphism into A. This implies that the unique function $A\to\{0\}$ is a categorical monomorphism. Hence monomorphisms in cotripleable categories need not be one-to-one!

4. **A characterization theorem.** The Lawvere-Linton theory of triples and theories (see, for example, [3] or [7]), applied to the dual of $\mathscr{S}$, yields the result that every cotripleable category $\mathscr{A}$ can be constructed as follows. There is a category π and a functor $T:\mathscr{S}\to\pi$ which is one-to-one on objects and preserves products and equalizers. $\mathscr{A}$ is equivalent to the category of functors $X:\pi\to\mathscr{S}$ such that XT is representable. Thus, an object of $\mathscr{A}$ is essentially a set A equipped, for each $\omega:n\to m$ in π, with a function $\omega_A:n^A\to m^A$, satisfying appropriate conditions; and $f:A\to B$ is a homomorphism in $\mathscr{A}$ if $m^f\omega_B=\omega_A n^f$. I.e., f is a homomorphism if, for each $\omega:n\to m$ and $h:B\to n$, we have $\omega_B(h)f=\omega_A(hf):A\to m$. In principle, this should give us a concept of theory for universal coalgebra exactly parallel to the one in universal algebra. But in practice this is not quite true; if we attempt to generate "free" theories we get coalgebra categories which are are not cotripleable but which are QCT. Thus, it is the QCT categories which turn out to have a natural description in terms of "operations and relations."

In what follows we shall presume familiarity with [7]. Also it will be necessary to manipulate proper classes, so we place ourselves in a set theory containing one of the usual mechanisms for doing so, e.g. universes. The category of sets is then an object of a category $\mathscr{C}$ of "classes" of "large sets."

THEOREM 4. *A category $\mathscr{A}$, equipped with a functor $U:\mathscr{A}\to\mathscr{S}$, is QCT iff there is a class of operations $\omega:n\to m$, for various sets n and m, subject to equality of various compositions $n\to m\to r$ and $n\to p\to r$, such that $\mathscr{A}$ is equivalent to the category of sets A equipped with operations $\omega_A:n^A\to m^A$, and of functions $f:A\to B$ satisfying $\omega_B(h)f=\omega_A(hf)$ for $h:B\to n$.*

PROOF. The proof that categories of the described form are QCT consists of lengthy but straightforward verifications and is omitted. Suppose that $U:\mathscr{A}\to\mathscr{S}$ is QCT. Then the composition $\mathscr{A}\to\mathscr{S}\to\mathscr{C}$ has a "coalgebraic closure," the category $\mathscr{B}$ of coalgebras in $\mathscr{C}$ over the density cotriple (G, ε, δ). And $\mathscr{B}$ can be described in terms of operations and relations as described above; if $V: \mathscr{B} \to \mathscr{C}$ is the forgetful functor then π can be obtained from the formula $\pi(n, m) = n.t.(n^V, m^V)$.

We claim that $\mathscr{A}$ is equivalent to the full subcategory of $\mathscr{B}$, whose objects are the G-algebras whose underlying classes are sets. Once this is shown, we need only observe that if A is a set, any $\omega_A: n^A \to m^A$ is determined by operations for which n and m are sets. Thus, let X be a G-algebra and also a set. Then there is $\xi: X \to GX$ such that $\varepsilon_X \xi = 1_X$ and $G(\xi)\xi = \delta_X \xi$: $X \to G^2X$. Here GX and G^2X are colimits of large diagrams in $\mathscr{A}$, and our method will be to replace these large diagrams by small ones, whose colimits will be in $\mathscr{A}$ since $\mathscr{A}$ is QCT. For each x in X choose A_x in $\mathscr{A}$, a_x in UA_x, and $f_x: UA_x \to X$ such that $\rho_{f_x}(a_x) = \xi(x)$. Here, if $f: UA \to X$, $\rho_f: UA \to GX$ comes from the construction of GX as a colimit. Let $\mathscr{D}$ be the category whose objects are pairs (A_x, f) where $f: UA_x \to X$, and a map $\varphi: (A_x, f) \to (A_y, g)$ is $\varphi: A_x \to A_y$ such that $fU(\varphi) = g$. The diagram $\mathscr{D} \to \mathscr{A}$, $(A_x, f) \mapsto A_x$, has a colimit D and there are induced maps $\sigma_f: A_x \to D$. Let $\mathscr{K}$ be the category whose objects are pairs (A_x, f) where $f: UA_x \to UD$ and maps defined as for $\mathscr{D}$. The diagram $\mathscr{K} \to \mathscr{A}$ has a colimit K and there are induced maps $\tau_f: A_x \to K$. We construct $z: X \to D$, $s: D \to X$, $t: K \to D$, and d and $k: D \to K$ as follows. Let $z(x) = \sigma_{f_x}(a_x)$, $s\sigma_f = f$, $d\sigma_f = \tau_{\sigma_f}$, $k\sigma_f = \tau_{zf}$, and $t\tau_g = g$. Then we have all the conditions for a split equalizer diagram, $zs = tk$, $sz = 1$, $td = 1$, *except* possibly $dz = kz$. For each x in X, $\delta_X \xi(x) = G(\xi)\xi(x)$ holds in GX, and thus, by the construction of colimits, in order that $dz(x) = kz(x)$ should hold in K we must adjoin some finite number of objects to the categories $\mathscr{D}$ and $\mathscr{K}$. We do this for each x and then redefine d, k, etc. Then X becomes a split equalizer of d and k. Since $\mathscr{A}$ is QCT, this implies that X is isomorphic to an object of $\mathscr{A}$. It remains to be verified that the inclusion $\mathscr{A} \to \mathscr{B}$ is full and faithful, but this follows by standard arguments. $\square$

It is unclear how to fit the examples of §§2 and 3 into this framework. At least, as an immediate consequence of Theorem 4, we can state that the set-theoretic image of a homomorphism in a QCT category is always a subobject. We conclude with an example showing that the same statement is not true for inverse images. Let $\mathscr{A}$ be the category of sets equipped with a single operation $\omega: 2 \to 2$, and functions $f: A \to B$ such that for each $C \subseteq B$, $f^{-1}\omega_B(C) = \omega_A f^{-1}(C)$. Let $A = \{0, 1, 2\}$, $B = \{0, 1\}$, $f(0) = f(1) = 0$, $f(2) = 1$, and $\omega_B = \mathrm{id}$. Then $\{0\}$ is a subobject of B. If f is to be a homomorphism, we must have $\omega_A(\varnothing) = \varnothing$, $\omega_A(\{0, 1\}) = \{0\}$, $\omega_A(\{2\}) = \{1\}$, $\omega_A(A) = A$. All other values of ω_A are arbitrary so we need only put $\omega_A(\{0\}) = \varnothing$, $\omega_A(\{0, 2\}) = \{0\}$ and then $f^{-1}(\{0\}) = \{0, 1\}$ is not a subobject of A; the inclusion is not a homomorphism.

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