A FACTORABLE WEIGHT WITH ZERO SZEGÖ INFIMUM

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Abstract. The associated Szegö infimum of a factorable operator valued weight function need not be nonzero. An example is constructed using algebraic properties of vectorial Toeplitz operators.

Let $N(\cdot)$ be an essentially bounded operator-valued function whose domain is the unit circle and whose range is in the space of bounded non-negative operators on a separable Hilbert space $\mathcal{H}$. As is well known, if $N(\cdot)$ admits a factorization $\theta(\cdot)\theta(\cdot)$ where $\theta(\cdot)$ is an outer function, then the Szegö infimum for a vector $c$ in $\mathcal{H}$ equals $\|\theta(0)c\|$ [3, p. 224]. We give an extremely simple example showing that it is possible for $N(\cdot)$ to be factorable but with associated Szegö infimum nontrivially equal to zero for some vector $c$ in $\mathcal{H}$. To state this example, it is more natural to use an algebraic Toeplitz model [2].

Let $\mathcal{H}$ be a separable Hilbert space with $S$ denoting a unilateral shift on $\mathcal{H}$ of infinite multiplicity. Set $\mathcal{C}=\ker S^*$ and define on $\mathcal{C}$ a unilateral shift $V_0$ of multiplicity one. By a diagonal matrix extend $V_0$ to an operator $V$ on $\mathcal{H}$. Let $A=V^*+S$ and note that $A$ is $S$-analytic and has trivial kernel. In addition, $A^*$ has trivial kernel so that $A$ is $S$-outer. Define the nonnegative $S$-Toeplitz operator $T=A^*A$ and consider the Szegö infimum relative to $T$ for the vector $c$ of norm one in the kernel of $V_0^*$. Computing, we have that

$$\inf_{f\in\mathcal{C}} \|T(c - Sf)\|^{1/2} = \inf_{f\in\mathcal{C}} \|A(c - Sf)\| = \inf_{f\in\mathcal{C}} \|Sc - SAf\| = 0,$$

since $\text{cl}(A\mathcal{C})=\mathcal{C}$. As was stated, the Szegö infimum for a vector $c$ in $\mathcal{C}$ computes $\|A_0(c)\|$ where $A_0=P_\mathcal{C}A|\mathcal{C}$. Thus the example constructed is one in which the kernel of $A_0$ is nontrivial but $\text{cl}(A^*\mathcal{C})=\mathcal{C}$. A necessary and sufficient condition for a positive Szegö infimum is the containment of any dense subset of the kernel of $S^*$ in the range of the nonnegative square root of the $S$-Toeplitz operator $T$ [1].

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