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A FACTORABLE WEIGHT WITH ZERO SZEGÖ INFIMUM

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ABSTRACT. The associated Szegö infimum of a factorable operator valued weight function need not be nonzero. An example is constructed using algebraic properties of vectorial Toeplitz operators.

Let $N(\cdot)$ be an essentially bounded operator-valued function whose domain is the unit circle and whose range is in the space of bounded nonnegative operators on a separable Hilbert space $\mathfrak{C}$. As is well known, if $N(\cdot)$ admits a factorization $\theta^*(\cdot)\theta(\cdot)$ where $\theta(\cdot)$ is an outer function, then the Szegö infimum for a vector c in $\mathfrak{C}$ equals $\|\theta(0)c\|$ [3, p. 224]. We give an extremely simple example showing that it is possible for $N(\cdot)$ to be factorable but with associated Szegö infimum nontrivially equal to zero for some vector c in $\mathfrak{C}$. To state this example, it is more natural to use an algebraic Toeplitz model [2].

Let $\mathfrak{H}$ be a separable Hilbert space with S denoting a unilateral shift on $\mathfrak{H}$ of infinite multiplicity. Set $\mathfrak{C}=\text{Ker } S^*$ and define on $\mathfrak{C}$ a unilateral shift V_0 of multiplicity one. By a diagonal matrix extend V_0 to an operator V on $\mathfrak{H}$. Let $A=V^*+S$ and note that A is S-analytic and has trivial kernel. In addition, A^* has trivial kernel so that A is S-outer. Define the nonnegative S-Toeplitz operator $T=A^*A$ and consider the Szegö infimum relative to T for the vector c of norm one in the kernel of V_0^*. Computing, we have that

$$\inf_{f\in\mathfrak{H}}\langle T(c-Sf),c-Sf\rangle^{1/2}=\inf_{f\in\mathfrak{H}}\|A(c-Sf)\|=\inf_{f\in\mathfrak{H}}\|Sc-SAf\|=0,$$

since $\text{cl}(A\mathfrak{H})=\mathfrak{H}$. As was stated, the Szegö infimum for a vector c in $\mathfrak{C}$ computes $\|A_0(c)\|$ where $A_0=P_{\mathfrak{C}}A|\mathfrak{C}$. Thus the example constructed is one in which the kernel of A_0 is nontrivial but $\text{cl}(A^*\mathfrak{H})=\text{cl}(A\mathfrak{H})=\mathfrak{H}=\sum_{j=0}^{\infty} S^j\text{cl}(A_0\mathfrak{C})$. A necessary and sufficient condition for a positive Szegö infimum is the containment of any dense subset of the kernel of S^* in the range of the nonnegative square root of the S-Toeplitz operator T [1].

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