NORMAL SUBGROUPS CONTAINED IN THE FRATTINI SUBGROUP

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Abstract. Let $H$ be a normal subgroup of the finite group $G$. If $H$ has a subgroup $K$ which is normal in $G$, satisfies $|K| > |K \cap Z_i(H)| = p$ and is not of nilpotence class 2, then $H$ is not contained in the Frattini subgroup of $G$.

All groups considered are finite. The ascending central series of a group $G$ is denoted by $1 = Z_0(G) \leq Z_1(G) \leq \cdots$. The Frattini subgroup of $G$, the intersection of all maximal proper subgroups, is denoted by $\Phi(G)$. If $Z_i(G) = G$ for some $i$, $G$ is called nilpotent and the smallest such $i$ is the nilpotence class of $G$, denoted by $\text{cl}(G)$.

E. L. Stitzinger [2] has stated the following result.

Theorem. Let $H$ be a $p$-group such that

(i) $|Z_1(H)| = p$;
(ii) there exists an abelian characteristic subgroup $A$ of $H$, $Z_1(H) < A \leq Z_2(H)$.

Then $H$ cannot be a normal subgroup contained in $\Phi(G)$ for any group $G$.

Unfortunately, there is a gap in the proof of Lemma 3 of [2], which is the basis for this theorem. We correct this oversight and prove a generalization of Stitzinger's theorem.

Lemma 3 of [2] says essentially this:

Lemma. Let $A$ be an abelian $p$-group with subgroup $\langle z \rangle$ of order $p$ such that $A / \langle z \rangle$ is elementary abelian. If $T = \{ \sigma \in \text{Aut}(A) | \sigma(\langle z \rangle) \in \langle z \rangle \}$ and $S = \{ \sigma \in T | \sigma(\langle z \rangle) = z \text{ and } \sigma(a)a^{-1} \in \langle z \rangle \text{ for each } a \in A \}$, then $S$ is complemented in $T$.

Presented to the Society, January 20, 1972; received by the editors October 12, 1971 and, in revised form, February 3, 1972.

AMS 1970 subject classifications. Primary 20D25; Secondary 20D15.

Key words and phrases. $p$-group, normal subgroup, Frattini subgroup.

1 Work supported by National Science Foundation grant GP-29041X.

2 NSF Science Faculty Fellow, 1970–1971. A portion of the content of this article is contained in this author's doctoral dissertation, written at the University of Cincinnati during 1970–1971 under the direction of Professor Donald Parker.

\$\$ American Mathematical Society 1972

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Let $x_1, x_2, \ldots, x_k \in A$ such that $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ is a basis for $A/\langle z \rangle$, where $\bar{x}_i = x_i/\langle z \rangle$. Then each $a \in A$ can be uniquely represented in the form $x_1^{a_1} \cdots x_k^{a_k} z^s$ with $0 \leq a_i, s < p$. Note that $a_i, s$ cannot be taken as elements of the field $F_p$ (e.g. take $A = \mathbb{C}$). The proof is as follows.

**Proof.** Let $B = \{x_1^{a_1} \cdots x_k^{a_k} | 0 \leq a_i < p, 1 \leq i \leq k\}$, and let

$$M = \{\sigma \in T \mid \sigma(B) = B\}.$$ 

$M$ is a subgroup of $T$ and $M \cap S$ is trivial. If $A$ is elementary abelian, then $M = \{\sigma \in T | \sigma(x_i) \in B, 1 \leq i \leq k\}$ and Stitzinger's argument is valid, so we assume that $A$ is not of exponent $p$.

Now $MS$ splits over $S$, an abelian (see Lemma 1 of [2]) $p$-group. By a result of Gaschütz [1, Kapitel I, Hauptsatz 17.4] it suffices to show that $[T: MS]$ is $p$-free.

There is an element $x \in A$ with $|x| = p^2$ and, with no loss of generality, $z = x^p$. Then $A = \langle x \rangle \times Y$ with $Y$ elementary abelian and we can take $x_1 = x$ and $\langle x_2, \ldots, x_k \rangle = Y$. Let $\pi : A \to Y$ be the projection homomorphism.

Let $\rho \in T$ be such that $\rho(a) a^{-1} \in \langle z \rangle Y$ for each $a \in A$. Define $\sigma : \{x\} \cup Y \to A$ by $\sigma(x) = \pi(\rho(x))$ and $\sigma(y) = \pi(\rho(y))$ for all $y \in Y$. $\sigma$ extends to an element of $T$ and in fact $\sigma \in M$. Since $\rho(a) a^{-1} \in \langle z \rangle Y$ for each $a \in A$, $\sigma(x) \equiv \rho(x) \mod(z)$ and $\sigma(y) \equiv \rho(y) \mod(z)$ for all $y \in Y$. Thus $\sigma^{-1} \rho \in S$ and it follows that $\rho \in MS$. Now $|A/\langle z \rangle Y| = p$ and so each $p$-element $\rho$ of $T$ satisfies $\rho(a) a^{-1} \in \langle z \rangle Y$ for all $a \in A$. Hence $MS$ contains all the $p$-elements of $T$ and the Lemma follows.

The arguments of [2], as corrected, are seen to suffice to prove Stitzinger's theorem with the weaker hypothesis that there exists an abelian characteristic subgroup $A$ of $H$ with $A \not\subseteq Z_2(H)$, $A \not\subseteq Z_1(H)$ and $|A \cap Z_1(H)| = p$. We use this stronger version to prove our main result.

**Theorem.** Let $H$ be a $p$-group with a characteristic subgroup $K$ such that

(i) $\text{cl}(K) \neq 2$;

(ii) $|K| > |K \cap Z_1(H)| = p$.

Then $H$ cannot be a normal subgroup contained in $\Phi(G)$ for any group $G$.

**Proof.** Let $A = Z_1(K) \cap Z_6(H)$. $A$ is a nontrivial characteristic subgroup of $H$, $A \not\subseteq Z_2(H)$ and $A \cap Z_1(H) = K \cap Z_1(H)$. If $A \cap Z_1(H) < A$, then the stronger version of Stitzinger's theorem implies the desired result. Assume otherwise, i.e. that $A \not\subseteq Z_1(K)$. Then $Z_1(K) \not\subseteq Z_1(H)$ for, if not, $Z_1(K)/Z_1(H)$ has nontrivial intersection with $Z_2(H)/Z_1(H)$. Thus, $|Z_1(K)| = p$. $K$ is therefore nonabelian and $\text{cl}(K) \geq 3$. $K' \subseteq Z_1(K)$ is an abelian characteristic subgroup of $K$, contained in $Z_2(K)$, and of order at least $p^2$. Hence, by Stitzinger's result, $K$ cannot be a normal subgroup.
contained in $\Phi(G)$ for any group $G$. As $K$ is a characteristic subgroup of $H$, the same is true of $H$.

REMARK. The assumption that the indicated subgroups are characteristic in $H$ can be replaced throughout by the assumption of normality in a fixed extension $G$ of $H$ $(H \unlhd G)$ with the conclusion that $H \unlhd \Phi(G)$.

REFERENCES


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