PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

NORMAL SUBGROUPS CONTAINED IN THE FRATTINI SUBGROUP[1]

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ABSTRACT. Let H be a normal subgroup of the finite group G. If H has a subgroup K which is normal in G, satisfies $|K|>|K\cap Z_1(H)|=p$ and is not of nilpotence class 2, then H is not contained in the Frattini subgroup of G.

All groups considered are finite. The ascending central series of a group G is denoted by $1=Z_0(G)\leqq Z_1(G)\leqq\cdots$. The Frattini subgroup of G, the intersection of all maximal proper subgroups, is denoted by $\Phi(G)$. If $Z_i(G)=G$ for some i, G is called nilpotent and the smallest such i is the nilpotence class of G, denoted by $\mathrm{cl}(G)$.

E. L. Stitzinger [**2**] has stated the following result.

THEOREM. *Let H be a p-group such that*

(i) $|Z_1(H)|=p$;

(ii) *there exists an abelian characteristic subgroup A of H, $Z_1(H)<A\leqq Z_2(H)$.*

Then H cannot be a normal subgroup contained in $\Phi(G)$ for any group G.

Unfortunately, there is a gap in the proof of Lemma 3 of [**2**], which is the basis for this theorem. We correct this oversight and prove a generalization of Stitzinger's theorem.

Lemma 3 of [**2**] says essentially this:

LEMMA. *Let A be an abelian p-group with subgroup $\langle z\rangle$ of order p such that $A/\langle z\rangle$ is elementary abelian. If $T=\{\sigma\in \mathrm{Aut}(A)|\sigma(z)\in\langle z\rangle\}$ and $S=\{\sigma\in T|\sigma(z)=z$ and $\sigma(a)a^{-1}\in\langle z\rangle$ for each $a\in A\}$, then S is complemented in T.*

Presented to the Society, January 20, 1972; received by the editors October 12, 1971 and, in revised form, February 3, 1972.

AMS 1970 *subject classifications.* Primary 20D25; Secondary 20D15.

Key words and phrases. p-group, normal subgroup, Frattini subgroup.

[1] Work supported by National Science Foundation grant GP-29041X.

[2] NSF Science Faculty Fellow, 1970–1971. A portion of the content of this article is contained in this author's doctoral dissertation, written at the University of Cincinnati during 1970–1971 under the direction of Professor Donald Parker.

Let $x_1, x_2, \cdots, x_k \in A$ such that $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k$ is a basis for $A/\langle z\rangle$, where $\bar{x}_i = x_i\langle z\rangle$. Then each $a \in A$ can be uniquely represented in the form $x_1^{a_1} \cdots x_k^{a_k} z^s$ with $0 \leqq a_i, s < p$. Note that a_i, s cannot be taken as elements of the field F_p (e.g. take $A = C_9$). The proof is as follows.

PROOF. Let $B = \{x_1^{a_1} \cdots x_k^{a_k} \mid 0 \leqq a_i < p, 1 \leqq i \leqq k\}$, and let

$$M = \{\sigma \in T \mid \sigma(B) = B\}.$$

M is a subgroup of T and $M \cap S$ is trivial. If A is elementary abelian, then $M = \{\sigma \in T \mid \sigma(x_i) \in B, 1 \leqq i \leqq k\}$ and Stitzinger's argument is valid, so we assume that A is not of exponent p.

Now MS splits over S, an abelian (see Lemma 1 of [**2**]) p-group. By a result of Gaschütz [**1**, Kapitel I, Hauptsatz 17.4] it suffices to show that $[T:MS]$ is p-free.

There is an element $x \in A$ with $|x| = p^2$ and, with no loss of generality, $z = x^p$. Then $A = \langle x\rangle \times Y$ with Y elementary abelian and we can take $x_1 = x$ and $\langle x_2, \cdots, x_k\rangle = Y$. Let $\pi: A \to Y$ be the projection homomorphism.

Let $\rho \in T$ be such that $\rho(a)a^{-1} \in \langle z\rangle Y$ for each $a \in A$. Define $\sigma: \{x\} \cup Y \to A$ by $\sigma(x) = x\pi(\rho(x))$ and $\sigma(y) = \pi(\rho(y))$ for all $y \in Y$. σ extends to an element of T and in fact $\sigma \in M$. Since $\rho(a)a^{-1} \in \langle z\rangle Y$ for each $a \in A$, $\sigma(x) \equiv \rho(x) \bmod \langle z\rangle$ and $\sigma(y) \equiv \rho(y) \bmod \langle z\rangle$ for all $y \in Y$. Thus $\sigma^{-1}\rho \in S$ and it follows that $\rho \in MS$. Now $|A/\langle z\rangle Y| = p$ and so each p-element ρ of T satisfies $\rho(a)a^{-1} \in \langle z\rangle Y$ for all $a \in A$. Hence MS contains all the p-elements of T and the Lemma follows.

The arguments of [**2**], as corrected, are seen to suffice to prove Stitzinger's theorem with the weaker hypothesis that there exists an abelian characteristic subgroup A of H with $A \leqq Z_2(H)$, $A \nleqq Z_1(H)$ and $|A \cap Z_1(H)| = p$. We use this stronger version to prove our main result.

THEOREM. *Let H be a p-group with a characteristic subgroup K such that*

(i) $\mathrm{cl}(K) \neq 2$;

(ii) $|K| > |K \cap Z_1(H)| = p$.

Then H cannot be a normal subgroup contained in $\Phi(G)$ for any group G.

PROOF. Let $A = Z_1(K) \cap Z_2(H)$. A is a nontrivial characteristic subgroup of H, $A \leqq Z_2(H)$ and $A \cap Z_1(H) = K \cap Z_1(H)$. If $A \cap Z_1(H) < A$, then the stronger version of Stitzinger's theorem implies the desired result. Assume otherwise, i.e. that $A \leqq Z_1(H)$. Then $Z_1(K) \leqq Z_1(H)$ for, if not, $Z_1(K)Z_1(H)/Z_1(H)$ has nontrivial intersection with $Z_2(H)/Z_1(H)$. Thus, $|Z_1(K)| = p$. K is therefore nonabelian and $\mathrm{cl}(K) \geqq 3$. $K' \cap Z_2(K)$ is an abelian characteristic subgroup of K, contained in $Z_2(K)$, and of order at least p^2. Hence, by Stitzinger's result, K cannot be a normal subgroup contained in $\Phi(G)$ for any group G. As K is a characteristic subgroup of H, the same is true of H.

REMARK. The assumption that the indicated subgroups are characteristic in H can be replaced throughout by the assumption of normality in a fixed extension G of H ($H \lhd G$)with the conclusion that $H \nleq \Phi(G)$.

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