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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PERTURBED LINEAR SYSTEMS

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ABSTRACT. The existence of solutions of the system $y'+Ay=f(t,y)$ having the form $y(t)=Z(t)a(t)$ is proved, where $Z(t)$ satisfies $Z'+AZ=0$ and the vector $a(t)$ has limit α as t increases. Estimates for the rate of convergence to zero of $a(t)-\alpha$ and of $y(t)-Z(t)\alpha$ are obtained.

Let $Z(t)$ be a fundamental matrix of solutions for

$$z' + Az = 0, \tag{1}$$

where z is an n-vector and A is an $n\times n$ matrix of constants. We shall be concerned with the possibility of writing solutions of

$$y' + Ay = f(t, y) \tag{2}$$

(y, f—n-vectors) in the form

$$y(t) = Z(t)a(t) \tag{3}$$

where the vector $a(t)$ has (finite) limit α as $t\to\infty$. Our conditions will be such that both $a(t)-\alpha$ and $y(t)-Z(t)\alpha$ converge to zero, and we obtain estimates on the rates of convergence.

Problems of this nature have been investigated ([**4**]–[**7**], [**10**]–[**13**]) for nth order differential equations; of these papers [**4**], [**5**], [**13**] assume that the linear differential equation corresponding to (1) is disconjugate, and are thus able to obtain more precise results. Only [**1**], [**2**], [**4**], [**9**] seem to have considered the more general question of systems. The present results are related to these; however, we are able to obtain results for a more general class of functions f (see Corollary 2 and the example preceding it), and our asymptotic estimates are better. The latter is accomplished, in part, by estimating separately each component of f and by restricting consideration to constant matrices A.

For an n-vector a we use the norm $\|a\|=\sum_{i=1}^{n}|a_i|$, we use also the notations $|a|=(|a_1|,|a_2|,\cdots,|a_n|)$ and $\mathbf{1}=(1,1,\cdots,1)$. For two n-vectors p, q we write $p\geqq q$ if $p_1\geqq q_1, p_2\geqq q_2,\cdots,p_n\geqq q_n$. For an $n\times n$

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matrix $B=(b_{ij})$ we shall form an n-vector $|||B|||$ with components $|||B|||_i=\sup_{1\leqq j\leqq n}|b_{ij}|$, and we shall denote by $|B|$ the matrix $(|b_{ij}|)$. Note that we have $|B\boldsymbol{a}|\leqq|||B|||\cdot\|\boldsymbol{a}\|$.

THEOREM. *Suppose for a given vector* $\boldsymbol{\alpha}$ *there exist* $T>0$, *a fundamental matrix* $Z(t)$ *of* (1), *and functions* g, ψ *with the following properties*:

(i) f *is continuous on* $[T,\infty)\times R^n$;

(ii) $g(t,s,x)$ *is continuous on* $[T,\infty)\times[T,\infty)\times R^n$, $0\leqq g(t,s,\boldsymbol{p})\leqq g(t,s,\boldsymbol{q})$ *whenever* $0\leqq\boldsymbol{p}\leqq\boldsymbol{q}$, *and* g *satisfies* $|Z^{-1}(t)f(s,y)|\leqq g(t,s,|y-Z(s)\boldsymbol{\alpha}|)$ *for* $(t,s,y)\in[T,\infty)\times[T,\infty)\times R^n$;

(iii) $\psi\in C([T,\infty))$, $\psi>0$, $\lim_{t\to\infty}\psi(t)=0$, *and*

$$\psi(t)\geqq\int_t^\infty g(s,s,|Z(s)|\,\psi(s))\,ds$$

on $[T,\infty)$.

Then there exist solutions y *of* (2) *of the form* (3) *where* $\lim_{t\to\infty}\boldsymbol{a}(t)=\boldsymbol{\alpha}$, *and we have*

$$|\boldsymbol{a}(t)-\boldsymbol{\alpha}|\leqq\psi(t), \tag{4}$$

$$|y(t)-Z(t)\boldsymbol{\alpha}|\leqq\int_t^\infty g(s-t,s,|Z(s)|\,\psi(s))\,ds \tag{5}$$

on $[T,\infty)$.

PROOF. Define an operator F on $\mathscr{A}\equiv\{x(t)\in C([T,\infty)):|x(t)-\boldsymbol{\alpha}|\leqq\psi(t)\text{ for }t\geqq T\}$ by

$$Fx(t)=\boldsymbol{\alpha}-\int_t^\infty Z^{-1}(s)f(s,Z(s)x(s))\,ds;$$

we shall use the Schauder-Tychonov theorem [2] to prove that F has a fixed point in $\mathscr{A}$. Since for $x\in\mathscr{A}$

$$\left|\int_t^\infty Z^{-1}(s)f(s,Z(s)x(s))\,ds\right|\leqq\int_t^\infty g(s,s,|Z(s)[x(s)-\boldsymbol{\alpha}]|)\,ds$$
$$\leqq\int_t^\infty g(s,s,|Z(s)|\,\psi(s))\,ds\leqq\psi(t),$$

F is indeed defined on $\mathscr{A}$ and maps $\mathscr{A}$ into $\mathscr{A}$. It is easily seen in a similar manner that F is continuous, i.e., if $x_n\in\mathscr{A}$ and $x_n\to x$ uniformly on compact subsets of $[T,\infty)$, then $Fx_n\to Fx$ uniformly on compact subsets of $[T,\infty)$. Since the functions in $\mathscr{A}$ are bounded, it remains only to show that the functions of $F\mathscr{A}$ are equicontinuous. Let $x\in\mathscr{A}$ and $t_1\geqq t_2\geqq T$; then

$$|Fx(t_1)-Fx(t_2)|\leqq\int_{t_2}^{t_1}|Z^{-1}(s)f(s,Z(s)x(s))|\,ds\leqq\int_{t_2}^{t_1}g(s,s,|Z(s)|\,\psi(s))\,ds,$$

which is small if $|t_1-t_2|$ is since g is continuous.

Let $\boldsymbol{a}(t) \in \mathscr{A}$ be a fixed point of F; then $\boldsymbol{y}(t)=Z(t)\boldsymbol{a}(t)$ is a solution of (2) on $[T, \infty)$, and (4) is established. To prove (5) observe that

$$\begin{aligned}|\boldsymbol{y}(t) - Z(t)\boldsymbol{\alpha}| &= \left|Z(t)\int_t^\infty Z^{-1}(s)\boldsymbol{f}(s, Z(s)\boldsymbol{a}(s))\,ds\right| \\ &\leqq \int_t^\infty |Z^{-1}(s-t)\boldsymbol{f}(s, Z(s)\boldsymbol{a}(s))|\,ds \\ &\leqq \int_t^\infty \boldsymbol{g}(s-t, s, |Z(s)|\,\boldsymbol{\psi}(s))\,ds.\end{aligned}$$

Two subcases of this theorem are of particular interest, and we discuss them as corollaries. The first deals with the case where $\|Z^{-1}(t)\boldsymbol{f}(t, Z(t)\boldsymbol{a}(t))\|$ is integrable for any bounded continuous $\boldsymbol{a}$.

Corollary 1. *Suppose for some fundamental matrix $Z(t)$ of* (1) *there exists a constant M and a continuous function $\boldsymbol{h}$ on* $[0, \infty)^{n+2}$ *such that*

(i) $0 \leqq \boldsymbol{h}(t, s, \boldsymbol{p}) \leqq \boldsymbol{h}(t, s, \boldsymbol{q})$ *whenever* $0 \leqq \boldsymbol{p} \leqq \boldsymbol{q}$;

(ii) $|Z^{-1}(t)\boldsymbol{f}(s, \boldsymbol{y})| \leqq \boldsymbol{h}(t, s, |\boldsymbol{y}|)$;

(iii) $\int_0^\infty \boldsymbol{h}(s, s, M|||Z(s)|||)\,ds < \infty$.

Then for all $\boldsymbol{\alpha} \in R^n$ such that $\|\boldsymbol{\alpha}\| < \frac{1}{2}M$ the hypotheses of the theorem are satisfied with

$$\boldsymbol{g}(t, s, \boldsymbol{p}) \equiv \boldsymbol{h}(t, s, \boldsymbol{p} + \tfrac{1}{2}M\,|||Z(s)|||),$$

$$\boldsymbol{\psi}(t) \equiv \int_t^\infty \boldsymbol{g}(s, s, \tfrac{1}{2}M\,|||Z(s)|||)\,ds.$$

Proof. It is necessary to verify (ii) and (iii) of the theorem. But

$$|Z^{-1}(t)\boldsymbol{f}(s, \boldsymbol{y})| \leqq \boldsymbol{h}(t, s, |\boldsymbol{y} - Z(s)\boldsymbol{\alpha}| + |||Z(s)||| \cdot \|\boldsymbol{\alpha}\|) \leqq \boldsymbol{g}(t, s, |\boldsymbol{y} - Z(s)\boldsymbol{\alpha}|),$$

verifying (ii). Choosing T so large that $\int_T^\infty \boldsymbol{h}(s, s, M|||Z(s)|||)\,ds < (M/2n)\mathbf{1}$, and observing that $\boldsymbol{g}$ is increasing in its last argument, we have

$$\boldsymbol{\psi}(t) = \int_t^\infty \boldsymbol{g}(s, s, \tfrac{1}{2}M\,|||Z(s)|||)\,ds = \int_t^\infty \boldsymbol{h}(s, s, M\,|||Z(s)|||)\,ds < \frac{M}{2n}\mathbf{1},$$

so $\|\boldsymbol{\psi}(t)\| < \frac{1}{2}M$, and

$$\int_t^\infty \boldsymbol{g}(s, s, |Z(s)|\,\boldsymbol{\psi}(s))\,ds \leqq \int_t^\infty \boldsymbol{g}(s, s, \tfrac{1}{2}M\,|||Z(s)|||)\,ds = \boldsymbol{\psi}(t).$$

We remark that the error estimate (5) can be replaced by the weaker but more convenient statement

$$|\boldsymbol{y}(t) - Z(t)\boldsymbol{\alpha}| \leqq \int_t^\infty \boldsymbol{g}(s-t, s, \tfrac{1}{2}M\,|||Z(s)|||)\,ds. \tag{6}$$

This corollary is connected with some results of [2], [4], [5], [7], [11], [12] when specialized to the nth order linear differential equation

$$x^{(n)} = f(t, x, x', \cdots, x^{(n-1)}) \tag{7}$$

where f satisfies $|f(t, x, \cdots, x^{(n-1)})| \leqq \sum_{i=0}^{n-1} g_i(t)|x^{(i)}|^{r_i}$ with $r_i > 0$ and

$$\int_0^\infty t^{n-1}(1 + t + \cdots + t^{n-i-1})^{r_i} g_i(t)\, dt < \infty \qquad (i = 0, \cdots, n-1). \tag{8}$$

Indeed, writing (7) as a first-order system in the usual way (cf. [2]), we obtain the result that for any $\boldsymbol{\alpha}$ there exists a solution x_α of (7) such that

$$|x_\alpha^{(j)} - \alpha_{j+1} - t\alpha_{j+2} - \cdots - (t^{n-j-1}/(n-j-1)!)\alpha_n|$$
$$\leqq \int_t^\infty (s-t)^j \left\{ \sum_{i=0}^{n-1} \|\alpha\|^{r_i} (1 + s + \cdots + s^{n-i-1})^{r_i} g_i(s) \right\} ds$$
$$(j = 0, \cdots, n-1).$$

We turn now to a different application of our main result. Consider the example of (2) given by the single first-order equation

$$y' = f(t, y) \equiv (1/t)(y-1)^2 + 1/t^3; \tag{9}$$

with $\alpha = 1$, $Z(t) = 1$, we have $g(t, s, |y - \alpha|) = (1/t)|y-1|^2 + 1/t^3$. Since $(1/t)|y(t) - 1|^2$ is not necessarily integrable even when $\lim_{t\to\infty} y(t) = 1$, Corollary 1 is not applicable, and neither are the results of [2], [4]. However, out main theorem can be applied with $\psi(t) = t^{-2+\varepsilon}$ for any $\varepsilon > 0$, as is readily seen, and from (5) we conclude that there exist solutions $y(t)$ of (9) such that $|y(t) - 1| \leqq 1/t^2$. In this example the right-hand side $\boldsymbol{f}$ has the form

$$\boldsymbol{f}(t, \boldsymbol{y}) = \boldsymbol{f}_1(t, \boldsymbol{y} - Z(t)\boldsymbol{\alpha}) + \boldsymbol{f}_2(t, \boldsymbol{y}), \tag{10}$$

where $\boldsymbol{f}_2$ $(= t^{-3})$ is integrable for bounded $\boldsymbol{y}$ but $\boldsymbol{f}_1$ is *not* integrable even for all $\boldsymbol{y}$ tending to $Z(t)\boldsymbol{\alpha}$. We generalize this situation in the following corollary.

COROLLARY 2. *Suppose for some fundamental matrix* $Z(t)$ *of* (1) *and some* $\boldsymbol{\alpha}$ *we have* (10) *valid, where there exist continuous functions* $\boldsymbol{h}_1$, $\boldsymbol{h}_2$ *on* $[0, \infty)^{n+2}$ *satisfying*

(i) $0 \leqq \boldsymbol{h}_i(t, s, \boldsymbol{p}) \leqq \boldsymbol{h}_i(t, s, \boldsymbol{q})$ *if* $0 \leqq \boldsymbol{p} \leqq \boldsymbol{q}$ $(i = 1, 2)$;

(ii) $|Z^{-1}(t)\boldsymbol{f}_2(s, \boldsymbol{y})| \leqq \boldsymbol{h}_2(t, s, |\boldsymbol{y}|)$, *with*

$$\int_0^\infty \boldsymbol{h}_2(s, s, |||Z(s)||| \cdot \|\boldsymbol{\alpha}\|)\, ds < \infty;$$

(iii) $|Z^{-1}(t)\boldsymbol{f}_1(s, \boldsymbol{y})| \leqq \boldsymbol{h}_1(t, s, |\boldsymbol{y} - Z(s)\boldsymbol{\alpha}|)$.

Suppose finally that there exists a continuous function $\boldsymbol{\phi}(t)>0$ *satisfying, for some* $\varepsilon>0$ *and all large* t, $\boldsymbol{\phi}(t)\to 0$ *as* $t\to\infty$,

$$\boldsymbol{\phi}(t) \geqq \int_t^\infty \boldsymbol{h}_1(s, s, (1+\varepsilon)\,|Z(s)|\,\boldsymbol{\phi}(s))\,ds,$$

$$\int_t^\infty \boldsymbol{h}_2(s, s, 2\,|||Z(s)|||\cdot\|\boldsymbol{\alpha}\|)\,ds \leqq \varepsilon\boldsymbol{\phi}(t).$$

Then the hypotheses of the theorem are satisfied for large T *with*

$$\boldsymbol{g}(t, s, \boldsymbol{p}) = \boldsymbol{h}_1(t, s, \boldsymbol{p}) + \boldsymbol{h}_2(t, s, \boldsymbol{p} + |||Z(s)|||\cdot\|\boldsymbol{\alpha}\|),$$

$$\boldsymbol{\psi}(t) = \boldsymbol{\phi}(t) + \int_t^\infty \boldsymbol{h}_2(s, s, 2\,|||Z(s)|||\cdot\|\boldsymbol{\alpha}\|)\,ds.$$

PROOF. Hypothesis (ii) of the theorem is clearly satisfied; we have only to verify (iii). But for sufficiently large t,

$$\begin{aligned}\int_t^\infty \boldsymbol{g}(s, s, |Z(s)|\,\boldsymbol{\psi}(s))\,ds &\leqq \int_t^\infty \boldsymbol{h}_1(s, s, (1+\varepsilon)\,|Z(s)|\,\boldsymbol{\phi}(s))\,ds \\ &\quad + \int_t^\infty \boldsymbol{h}_2(s, s, |Z(s)|\,\boldsymbol{\psi}(s) + |||Z(s)|||\cdot\|\boldsymbol{\alpha}\|)\,ds \\ &\leqq \boldsymbol{\phi}(t) + \int_t^\infty \boldsymbol{h}_2(s, s, 2\,|||Z(s)|||\cdot\|\boldsymbol{\alpha}\|)\,ds = \boldsymbol{\psi}(t).\end{aligned}$$

As an application of Corollary 2 we determine conditions on the coefficients of the differential equation

$$u^{(n)} = \sum_{i=0}^{n-1} a_i(t)(u^{(i)})^{r_i} + b(t) \tag{11}$$

($r_i>1$) which guarantee the existence of a solution $u(t)$ such that $u^{(j)}(t)\to 0$ as $t\to\infty$ ($j=0, 1, \cdots, n-1$). If b and all the a_i are integrable, this follows from Corollary 1, hence we assume that not all a_i are integrable. Writing the equation as a system in the usual manner, we have

$$Z(t) = \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ 0 & 1 & t & \cdots & t^{n-2}/(n-2)! \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

for fundamental matrix and $\alpha=0$. With

$$h_{2,k}(t, s, |y|) = \frac{t^{n-k}}{(n-k)!}|b(s)| \qquad (k = 1, \cdots, n),$$

$$h_{1,k}(t, s, |y|) = \frac{t^{n-k}}{(n-k)!}\sum_{i=0}^{n-1}|a_i(s)|\,|y_{i+1}|^{r_i}$$

we have that $\boldsymbol{\phi}$ must satisfy

$$\phi_k(t) \geqq \int_t^\infty \frac{s^{n-k}}{(n-k)!}\sum_{i=0}^{n-1}|a_i(s)|\,(1+\varepsilon)^{r_i}\left[\sum_{j=i}^{n}\frac{s^{j-i}}{(j-i)!}\phi_j(s)\right]^{r_i} ds.$$

For convenience let $C_i=\sum_{j=1}^n ((n-1)!/(j-1)!\,(n-j)!)$ and $r=\max r_i$. Let $\phi_1(t)$ be any solution of

$$(12) \qquad \phi_1(t) \geqq (1+\varepsilon)^r \int_t^\infty \frac{s^{n-1}}{(n-1)!}\sum_{i=0}^{n-1}|a_i(s)|\left[\frac{C_i}{s^{i-1}}\phi_1(s)\right]^{r_i} ds$$

and define

$$\phi_j(t) = \frac{(n-1)!}{(n-j)!\,t^{j-1}}\phi_1(t) \qquad (j = 2, \cdots, n).$$

Then

$$\phi_j(t) \geqq (1+\varepsilon)^r \int_t^\infty \frac{s^{n-j}}{(n-j)!}\sum_{i=0}^{n-1}|a_i(s)|\left[\frac{C_i}{s^{i-1}}\phi_1(s)\right]^{r_i} ds$$

$$\geqq \int_t^\infty \frac{s^{n-j}}{(n-j)!}\sum_{i=0}^{n-1}|a_i(s)|\,(1+\varepsilon)^{r_i}\left[\sum_{k=i}^{n}\frac{s^{k-i}}{(k-i)!}\phi_k(s)\right]^{r_i} ds,$$

as required. It thus remains to show that (12) has a suitable solution. Suppose $0\leqq\phi_1(t)\leqq 1$, and let $q=\min r_i>1$, so $\phi_1^{r_i}\leqq\phi_1^q$; then if ϕ_1 satisfies

$$(13) \quad \phi_1(t) \geqq \int_t^\infty \frac{(1+\varepsilon)^r}{(n-k)!}s^{n-1}\sum_{i=0}^{n-1}|a_i(s)|\left(\frac{C_i}{s^{i-1}}\right)^{r_i}\phi_1^q(s)\,ds \equiv \int_t^\infty k(s)\phi_1^q(s)\,ds,$$

it will also satisfy (12). The equality in (13) can be solved to yield

$$\phi_1(t) = \left\{[\phi_1(t_0)]^{1-q} + (q-1)\int_{t_0}^t k(s)\,ds\right\}^{-1/(q-1)}$$

provided $\int^\infty k(s)\,ds=\infty$ and the positive constant $\phi_1(t_0)$ is chosen less than 1. Thus (11) has solutions tending to zero with their first $n-1$ derivatives if

$$\int^t \sum_{i=0}^{n-1}|a_i(s)|\,s^{r_i(1-i)+n-1}\,ds \to \infty$$

as $t\to\infty$ and $b(t)$ satisfies $\int_t^\infty s^{n-1}|b(s)|\,ds\leqq\varepsilon(n-1)!\,\phi_1(t)$ for some $\varepsilon>0$.

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