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# SURFACES WITH MAXIMAL LIPSCHITZ-KILLING CURVATURE IN THE DIRECTION OF MEAN CURVATURE VECTOR

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ABSTRACT. $M^2$ is an oriented surface in $E^{2+N}$. If $M^2$ is pseudo-umbilical, the Lipschitz-Killing curvature takes maximum in the direction of mean curvature vector. The converse is also investigated. Furthermore assuming that $M^2$ is closed, pseudo-umbilical and its Gaussian curvature has some nonnegative lower bound, $M^2$ is completely determined by the $M$-index of $M^2$.

1. Let $M^2$ be an oriented Riemannian surface with an isometric immersion $x:M^2\to E^{2+N}$ in a euclidean space $E^{2+N}$. Let $F(M^2)$ and $F(E^{2+N})$ be the bundles of orthonormal frames of $M^2$ and $E^{2+N}$ respectively. Throughout this paper we assume that the *mean curvature vector $H$ of $M^2$ is nowhere zero*. Let $B$ be the set of elements $b=(p, e_1, e_2, \cdots, e_{2+N})$ such that $(p, e_1, e_2)\in F(M^2)$, $e_3=H/|H|$ and that $(x(p), e_1, e_2, e_3, \cdots, e_{2+N})\in F(E^{2+N})$ whose orientation is coherent with that of $E^{2+N}$, identifying $e_i$ with $dx(e_i)$, $i=1, 2$. Let $\tilde{x}:B\to F(E^{2+N})$ be the mapping naturally defined by $\tilde{x}(b)=(x(p), e_1, \cdots, e_{2+N})$.

We have the differential forms $\omega_i$, $\omega_{ij}$, $\omega_{i\alpha}$, $\omega_{\alpha\beta}$ $(1\leqq i,j\leqq 2, 3\leqq\alpha, \beta\leqq 2+N)$ on $B$ derived from the basic forms and the connection forms on $F(E^{2+N})$ through $\tilde{x}$ as follows.

$$dx = \omega_1 e_1 + \omega_2 e_2, \qquad de_A = \sum_{B=1}^{2+N} \omega_{AB} e_B, \qquad \omega_{AB} = -\omega_{BA} \qquad (A, B = 1, 2, \cdots, 2+N);$$

$$\omega_{i\alpha} = \sum_{j=1}^{2} A_{\alpha ij}\omega_j, \qquad A_{\alpha ij} = A_{\alpha ji}.$$

In the following, for the summation notations $\sum_i$, $\sum_\alpha$ and $\sum_r$ we mean $\sum_{i=1}^2$, $\sum_{\alpha=3}^{2+N}$ and $\sum_{r=4}^{2+N}$, for the indices $i, j, \alpha, \beta, r, t$ we mean $1\leqq i,j\leqq 2$, $3\leqq\alpha, \beta\leqq 2+N$, $4\leqq r, t\leqq 2+N$.

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Now we choose $e_1$, $e_2$ as the principal directions of $e_3$, then with respect to the frame $(e_1, e_2, e_3, \cdots, e_{2+N})$ the matrices $A_\alpha=(A_{\alpha ij})$ are written in

$$(1)\qquad (A_{3ij}) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad (A_{rij}) = \begin{pmatrix} c_r & d_r \\ d_r & -c_r \end{pmatrix}.$$

$H=\frac{1}{2}(a+b)e_3$. When $a=b$, $M^2$ is *pseudo-umbilical*. Let $e$ be a unit normal vector to the tangent plane $dx(T_p(M^2))$ at $x(p)$. Then

$$(2)\qquad e = \sum_\alpha \xi_\alpha e_\alpha, \qquad \sum_\alpha \xi_\alpha^2 = 1.$$

Let $A(e)$ be the following matrix

$$A(e) = \sum_\alpha \xi_\alpha A_\alpha = \begin{pmatrix} a\xi_3 + \sum_r c_r\xi_r & \sum_r d_r\xi_r \\ \sum_r d_r\xi_r & b\xi_3 - \sum_r c_r\xi_r \end{pmatrix}.$$

The *Lipschitz-Killing curvature* $G(p, e)$ is given by [4]

$$(3)\qquad \begin{aligned} G(p, e) &= \det(A(e)) \\ &= ab\xi_3^2 + (b-a)\xi_3 \sum_r c_r\xi_r - \Big(\sum_r c_r\xi_r\Big)^2 - \Big(\sum_r d_r\xi_r\Big)^2. \end{aligned}$$

Let $S_2$ be the set of all real symmetric square matrices of order 2. Let $m: S_2 \to R$ be a linear transformation defined by [6] $m(A)=\frac{1}{2}$ trace $A$, $A \in S_2$. We denote the normal space to $x(M^2)$ at $x(p)$ by $N_p$, $N_p=\{X, X=\sum_\alpha \eta_\alpha e_\alpha, \eta_\alpha \in R\}$, and define a linear mapping $\bar{m}: N_p \to R$ by

$$(4)\qquad \bar{m}(X) = \sum_\alpha \eta_\alpha m(A_\alpha), \qquad X = \sum_\alpha \eta_\alpha e_\alpha.$$

The kernel of $\bar{m}$ is denoted by ker $\bar{m}$.

At any point $p \in M^2$ we take a frame $b=(p, e_1, \cdots, e_{2+N}) \in B$. Let $\psi_b: N_p \to S_2$ be the linear mapping defined by

$$(5)\qquad \psi_b\Big(\sum_\alpha \eta_\alpha e_\alpha\Big) = \sum_\alpha \eta_\alpha A_\alpha.$$

The dimension of $\psi_b(\ker \bar{m})$ is called the *M-index of* $M^2$ at $p$ and is denoted by $M$-index$_p$ $M^2$ [6].

2. We prove the following lemma.

LEMMA. *If at any point $p \in M^2$ the Lipschitz-Killing curvature $G(p, e)$ has maximum in the direction of $H$ then $ab \geq 0$ where $a$, $b$ are given by* (1).

PROOF. $e_3=H/|H|$, which is given by $\xi_3=1$, $\xi_r=0$ in (2). By (2) and (3) it is easy to see that if $G(p, e)$ takes maximum at $\xi_3=1$, $\xi_r=0$ then $(a-b)c_r=0$. Hence by (3) the maximum of $G(p, e)$ is $ab$. Now let $S_p$ be an arbitrary chosen unit circle in $N_p$ and $e'$ be a fixed point in $S_p$. Put $S_p^*=S_p-\{e'\}$. We choose $e \in S_p^*$ and $e_1(e)$, $e_2(e)$ as two unit orthogonal tangent vectors in the principal directions of $e$ and move $e$ differentiably on $S_p^*$. Then the principal curvatures $k_1(e)$ and $k_2(e)$ with respect to $e_1(e)$ and $e_2(e)$ are continuous on $S_p^*$. Now suppose $k_1(e)\neq 0$ at some $e \in S_p^*$. Then by the continuity of $k_1$ on $S^*$ and the fact $k_1(-e)=-k_1(e)$ we see that $k_1=0$ for some points in $S_p^*$. This implies that the Lipschitz-Killing curvature $G(p, e)=0$ for some $e \in S_p^*$. Since $ab$ is the maximum of $G(p, e)$ we conclude that $ab\geqq 0$. This is true for all $p \in M^2$.

3. If $M^2$ is pseudo-umbilical then $a=b$ and by (3) we have that $G(p, e)$ takes maximum in $e_3$. To get further results we consider the normal curvature $R^\alpha_{\beta ij}$ and scalar normal curvature $K_N$ [1]:

$$
(6)\qquad \begin{aligned} R^\alpha_{\beta kl} &= \sum_i (A_{\alpha ik}A_{\beta il} - A_{\alpha il}A_{\beta ik}),\\ K_N &= \sum_{\alpha,\beta,i,j} (A_{\alpha ik}A_{\beta jk} - A_{\alpha jk}A_{\beta ik})^2. \end{aligned}
$$

THEOREM 1. *At points $p$ with $M$-index$_p$ $M^2\geqq 2$, $M^2$ is pseudo-umbilical if and only if $G(p, e)$ has maximum in $e_3$; at points $p$ with $M$-index$_p$ $M^2=1$, $M^2$ is pseudo-umbilical if and only if $G(p, e)$ has maximum in $e_3$ and $K_N=0$; at points $p$ with $M$-index$_p$ $M^2=0$, $M^2$ is pseudo-umbilical if and only if $M^2$ is totally umbilical.*

PROOF. Suppose $M$-index$_p$ $M^2\geqq 2$. Otsuki in [6] showed that $M$-index$_p$ $M^2\leqq 2$, so $M$-index$_p$ $M^2=2$. Since $m(A_3)=\frac{1}{2}(a+b)\neq 0$ and $m(A_r)=0$ we have $\ker \bar{m}=\{\sum_r \eta_r e_r,\ \eta_r \in R\}$, $e_r \in \ker \bar{m}$ and $\psi_b(e_r)=A_r$. Hence for at least one $r$, $c_r\neq 0$. But $G(p, e)$ takes maximum in $e_3$; we have $(a-b)c_r=0$. Hence $a=b$ and $M^2$ is pseudo-umbilical. Next suppose $M$-index$_p$ $M^2=1$, $G(p, e)$ has maximum in $e_3$ and $K_N=0$. $A_r$ are given in (1). $\psi_b(\ker \bar{m})=\{\sum_r \eta_r A_r,\ \eta_r \in R\}$. If dim $(\psi_b(\ker \bar{m}))=M$-*index*$_p$ $M^2=1$ then there is $k$ so that $d_r=kc_r$ for any $r$. We have then $A_rA_t=A_tA_r$ and

$$K_N = 2\sum_{\beta,i,j} (A_{3ik}A_{\beta jk} - A_{3jk}A_{\beta ik})^2.$$

$K_N=0$ implies $A_3A_r=A_rA_3$, that is $(a-b)d_r=0$. If all $d_r=0$ then at least one $c_r\neq 0$ because $M$-index$_p$ $M^2=1$. Thus $G(p, e)$ having maximum at $e_3$ implies $a=b$. The inverse is clear. Finally suppose $M$-index $M^2=0$, then $c_r=d_r=0$, i.e., $A_r=0$. It is clear that $M^2$ is pseudo-umbilical, i.e., $a=b$, if and only if $M^2$ is totally umbilical.

4. In this section we assume that $M^2$ is a closed surface. For a symmetric matrix $A=(a_{ij})$ if we write $N(A)=\sum_{i,j} a_{ij}^2$, then we have $K_N=\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha)$. Chen in [1] proved the following results: The Veronese surface is the only closed pseudo-umbilical surface in euclidean space with parallel normal curvature and scalar normal curvature $K_N \neq 0$, the 2-sphere and the Clifford torus are the only closed pseudo-umbilical surfaces in euclidean space with scalar normal curvature $K_N=0$ and scalar curvature $R \geqq 0$. From these results we have the following two theorems.

THEOREM 2. *If $M^2$ is closed, pseudo-umbilical, $M$-index$_p$ $M^2=1$ for any $p$ and the Gaussian curvature $G(p) \geqq 0$ everywhere then $M^2$ is either a sphere or a Clifford torus.*

PROOF. If $M^2$ is pseudo-umbilical and $M$-index$_p$ $M^2=1$ we have by Theorem 1 that $K_N=0$. The scalar curvature $R=2G \geqq 0$ by assumption. By Chen's result $M^2$ is either a sphere or a Clifford torus.

THEOREM 3. *If $M^2$ is closed, pseudo-umbilical, $M$-index$_p$ $M^2=2$ for any $p$ and the Gaussian curvature $G(p) \geqq (N-2)a^2/2N-3$ everywhere then $M^2$ is a Veronese surface.*

PROOF. $M$-index$_p$ $M^2=2$ implies that $K_N \neq 0$. For the Laplacian of $A_{\alpha ij}$ in the case of pseudo-umbilical $M^2$ we have the known equality [2]:

$$\sum_{\alpha,i,j} A_{\alpha ij}\Delta A_{\alpha ij} = 2a\Delta a + 2a^2 S - \sum_{\alpha} S_\alpha^2 - \sum_{\alpha\neq\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) \tag{7}$$

where $S_\alpha=\sum_{i,j} A_{\alpha ij}^2=N(A_\alpha)$, $S=\sum_\alpha S_\alpha$. Since $M^2$ is pseudo-umbilical we have $A_3=aI$ and $\sum_{i,j} A_{3ij}\Delta A_{3ij}=2a\Delta a$. It is known also that [3] $N(A_\alpha A_\beta - A_\beta A_\alpha) \leqq 2N(A_\alpha)N(A_\beta)$. So we have $\sum_{r\neq t} N(A_r A_t - A_t A_r) \leqq 2\sum_{r\neq t} N(A_r)N(A_t) = 2\sum_{r\neq t} S_r S_t$. Let $\bar{S}=\sum_r S_r$ and noticing that $2a^2 S_3 = 2a^2(2a^2)=4a^4=S_3^2$ we have from (7):

$$\begin{aligned} \sum_{r,i,j} A_{rij}\Delta A_{rij} &\geqq 2a^2\bar{S} - \sum_r S_r^2 - 2\sum_{r\neq t} S_r S_t \\ &= 2a^2\bar{S} - \left(\sum_r S_r\right)^2 - 2\sum_{r<t} S_r S_t. \end{aligned} \tag{8}$$

Let $\sigma_1$, $\sigma_2$ be such that $(N-1)\sigma_1=\sum_r S_r=\bar{S}$, $(\tfrac{1}{2})(N-1)(N-2)\sigma_2=\sum_{r<t} S_r S_t$; it can easily be seen that [3]: $(N-1)^2(N-2)(\sigma_1^2-\sigma_2)=\sum_{r<t}(S_r-S_t)^2 \geqq 0$. Hence $\sigma_1^2 \geqq \sigma_2$ or

$$2\sum_{r<t} S_r S_t \leqq (N-2)\bar{S}^2/(N-1). \tag{9}$$

By (8) and (9) we have

$$\sum_{r,i,j} A_{rij}\Delta A_{rij} \geqq -(2N-3)\bar{S}^2/(N-1) + 2a^2\bar{S}. \tag{10}$$

The Gaussian curvature of $M^2$ is $G(p)=\sum_\alpha \det(A_\alpha)=a^2-\sum_r (c_r^2+d_r^2)=a^2-(\frac{1}{2})\bar{S}$. Therefore $\bar{S}=2(a^2-G)$. By (10) we have

$$-\sum_{r,i,j} A_{rij}\,\Delta A_{rij} \leqq 2\bar{S}[(2N-3)(a^2-G)/(N-1)-a^2].$$

Thus if $(2N-3)(a^2-G)/(N-1)-a^2\leqq 0$ or $G\geqq(N-2)a^2/(2N-3)$ then $\sum_{r,i,j} A_{rij}\Delta A_{rij}\geqq 0$. Now from the equality

$$(\tfrac{1}{2})\Delta\sum_{r,i,j}(A_{rij})^2 = \sum_{r,i,j,k}(A_{rijk})^2 + \sum_{r,i,j} A_{rij}\,\Delta A_{rij}, \tag{11}$$

where $A_{rijk}=A_{rij,k}$ (covariant derivative), we have that if $G\geqq(N-2)a^2/(2N-3)$ then $\Delta\sum_{r,i,j}(A_{rij})^2\geqq 0$. Since $M^2$ is closed we have $\Delta\sum_{r,i,j}(A_{rij})^2=0$. By (11) it implies $A_{rij,k}=0$. By (6) we have that $R^\alpha_{\beta kl}$ (noticing that $R^3_{\beta kl}\equiv 0$) is parallel and that $K_N$ is constant and $K_N\neq 0$. By Chen's result we conclude that $M^2$ is a Veronese surface.

THEOREM 4. *If $M^2$ is closed, $G(p, e)$ takes maximum in $e_3$ and $M$-index$_p$ $M^2=0$ everywhere then $M^2$ is embedded as a convex surface in an $E^3$.*

PROOF. $M$-index$_p$ $M^2=0$ implies that $c_r=d_r=0$. Then we may write $G(p,e)=ab\cos^2\theta_3$, $0\leqq\theta_3\leqq\pi$. By the lemma $ab\geqq 0$. The Gaussian curvature $G(p)$ in this case is $G(p)=\sum_\alpha\det(A_\alpha)=ab$. So we have $G(p)\geqq G(p,e)$ for all $e$. By the Gauss-Bonnet formula $\int_{M^2} G(p)\,dV=\int_{M^2} ab\,dV=4\pi(1-g)$. On the other hand the total curvature $K^*(p)$ of $M^2$ at $p$ is

$$K^*(p)=\int_{S^{N-1}}|G(p,e)|\,d\sigma_{N-1}=\int_{S^{N-1}} ab\cos^2\theta_3\,d\sigma_{N-1}=G(p)c_{N+1}/2\pi$$

where $S^{N-1}$ is the unit sphere in $E^N$ and $c_{N+1}$ is the volume of the unit sphere in $E^{2+N}$.

By a result due to Chern-Lashof [5] we have $(1/c_{N+1})\int_{M^2}K^*(p)\,dV\geqq 2+2g$, the equality sign holds if and only if $M^2$ is embedded as a convex surface in an $E^3$. On the other hand

$$\begin{aligned}(c_{N+1}/2\pi)4\pi(1-g) &= (c_{N+1}/2\pi)\int_{M^2} G(p)\,dV \\ &= \int_{M^2} K^*(p)\,dV \geqq c_{N+1}(2+2g).\end{aligned}$$

That is $1-g\geqq 1+g$. Thus it is necessary that $g=0$ and the equality sign holds. Hence $M^2$ is embedded as a convex surface in $E^3$.

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