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# A NOTE ON HIGHER DERIVATIONS AND INTEGRAL DEPENDENCE

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ABSTRACT. In this note we prove the following: THEOREM. *Let $R'$ be an associative commutative ring with identity. Suppose $R'$ is an integral extension of $R$, and $\delta=\{\delta_i\}$ is a higher derivation on $R'$ which restricts to a higher derivation on $R$. Suppose $p$ is a prime ideal in $R$ which is differential under $\delta$. Then there exists a prime ideal $p'$ in $R'$ such that $p'$ is $\delta$-differential and $p'\cap R=p$.*

**Introduction.** In this paper, we assume all rings are associative, commutative and have an identity. A subring of a given ring is assumed to have the same identity as the given ring.

Let $R$ be a ring. A higher derivation $\delta=\{\delta_q\}$ (of infinite rank) on $R$ is an infinite sequence of maps $\delta_q:R\to R$, $q=1, 2, 3, \cdots$, such that

(a) each $\delta_q$ is an additive group homomorphism;

(b) for all $x, y \in R$ and $q\geqq 1$,

$$\delta_q(xy) = x\delta_q(y) + \delta_1(x)\delta_{q-1}(y) + \cdots + \delta_{q-1}(x)\delta_1(y) + \delta_q(x)y \quad \text{(Leibnitz's rule).}$$

We shall abbreviate the last equation by writing $\delta_q(xy)=\sum_{i+j=q}\delta_i(x)\delta_j(y)$. We note that (a) and (b) imply that $\delta_q(1)=0$ for all $q$. Thus if $Z_0$ denotes the prime subring of $R$, i.e. the subring of $R$ generated by the identity element 1, then $\delta_q(Z_0)=0$ for all $q\geqq 1$.

If $\delta$ is a higher derivation on $R$ and $A$ is an ideal in $R$, then we shall say $A$ is $\delta$-differential if $\delta_q(A)\subset A$ for all $q\geqq 1$.

The purpose of this paper is to prove the following:

THEOREM 1. *Let $R'$ be a ring containing $R$. Suppose $R'$ is an integral extension of $R$, and $\delta$ is a higher derivation on $R'$ which restricts to a higher derivation on $R$. Suppose $p$ is a prime ideal in $R$ which is differential under $\delta$. Then there exists a prime ideal $p'$ in $R'$ such that $p'$ is $\delta$-differential and $p'\cap R=p$.*

This theorem appears as Theorem 2 in S. Sato's *On rings with a higher derivation* [2]. In his proof of this result, Sato seems to assume $R'$ is

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Noetherian. If we assume $R'$ is not Noetherian, then there is a gap in Sato's theorem which we shall fill in this note.

PROOF OF THEOREM 1. Following the first paragraph of Sato's proof, we can assume without loss of generality that $R$ is a quasi-local ring with maximal ideal $p$. Since $R'$ is an integral extension of $R$, $pR'\neq R'$. Clearly $pR'$ is a $\delta$-differential ideal in $R'$. Hence by [2, Theorem 1] there exists a maximal $\delta$-differential ideal $p'$ in $R'$ which contains $pR'$. If $p'$ is a prime ideal, then the theorem is complete. Hence we wish to prove that $p'$ is a prime ideal in $R'$.

Consider $(R')^-=R'/p'$. Since $p'\cap R=p$, $(R')^-$ is an integral extension of the field $\bar{R}=R/p$. Since both $p$ and $p'$ are $\delta$-differential ideals, $\delta$ induces a higher derivation $\bar{\delta}$ on $(R')^-$ which restricts to a higher derivation on $\bar{R}$. Specifically, for all $q\geqq 1$, $\bar{\delta}_q(r+p')=\delta_q(r)+p'$ for $r\in R'$. Note that $(R')^-$ has no proper $\bar{\delta}$-differential ideals. Otherwise $p'$ would not be a maximal $\delta$-differential ideal in $R'$.

Now let $N$ be any maximal ideal in $(R')^-$. If $N\neq 0$, then $\bar{\delta}(N)\not\subset N$. Hence there exists a nonzero element $x\in N$ such that $\bar{\delta}_q(x)\notin N$ for some $q\geqq 1$. Since $(R')^-$ is an integral extension of $\bar{R}$, $x$ satisfies some monic polynomial $f(X)\in\bar{R}[X]$. Let $f(X)=X^n+r_{n-1}X^{n-1}+\cdots+r_1X+r_0$. Since $\bar{R}$ is a field, $f(x)=0$ implies $r_0\in N$. Hence $r_0=0$. So $f(X)=X^n+r_{n-1}X^{n-1}+\cdots+r_1X$. We first argue that some $r_i$, $i=1,\cdots,n-1$, is not zero. That is, $x$ cannot be nilpotent. This follows from the following lemma:

LEMMA 1. *If $x^n=0$, then $\bar{\delta}_q(x)\in N$ for all $q\geqq 1$.*

PROOF. This lemma is argued via induction on $q$. Suppose $\bar{\delta}_1(x)\notin N$. Then we shall show that $n$ is bigger than every positive integer. Since $x\neq 0$, $n>1$. Since $\bar{\delta}_2(x^n)=0$, we get $\bar{\delta}_1(x^{n-1})\in N$. Hence $n>2$. Assume we have shown $n>m\geqq 2$. Successively applying $\bar{\delta}_2,\cdots,\bar{\delta}_m$ to the equation $x^n=0$, we get $\bar{\delta}_1(x^{n-1})\in N$, $\bar{\delta}_2(x^{n-1})$ and $\bar{\delta}_1(x^{n-2})\in N,\cdots,\bar{\delta}_{m-1}(x^{n-1})$ through $\bar{\delta}_1(x^{n-(m-1)})\in N$. Now

$$0=\bar{\delta}_{m+1}(x^n)=\sum_{i+j=m+1}\bar{\delta}_i(x)\bar{\delta}_j(x^{n-1}).$$

Thus, $\bar{\delta}_m(x^{n-1})\in N$. But $\bar{\delta}_m(x^{n-1})=\sum_{i+j=m}\bar{\delta}_i(x)\bar{\delta}_j(x^{n-2})$. Therefore $\bar{\delta}_{m-1}(x^{n-2})\in N$. By expanding this further, we get $\bar{\delta}_1(x^{n-m})\in N$. Hence $n>m+1$. Thus $\bar{\delta}_1(x)\in N$.

Assume we have shown that $\bar{\delta}_1(x),\cdots,\bar{\delta}_{q-1}(x)\in N$. If $\bar{\delta}_q(x)\notin N$, we shall again show that $n$ is bigger than every positive integer. The procedure is similar to the case $q=1$. Applying $\bar{\delta}_{2q}$ to the equation $x^n=0$ we get $\bar{\delta}_q(x^{n-1})\in N$. So $n>2$. Applying $\bar{\delta}_{2q+1},\cdots,\bar{\delta}_{3q}$ to $x^n=0$, we get $\bar{\delta}_{q+1}(x^{n-1}),\cdots,\bar{\delta}_{2q}(x^{n-1})$ are all in $N$. But

$$\bar{\delta}_{2q}(x^{n-1})=\sum_{i+j=2q}\bar{\delta}_i(x)\bar{\delta}_j(x^{n-2})$$

being an element of $N$ implies $\bar{\delta}_q(x^{n-2}) \in N$. Thus $n>3$. I think it is clear now how one proceeds by induction to show $n>m$ for any $m$. Thus $\bar{\delta}_q(x) \in N$, and the proof of Lemma 1 is complete. □

Lemma 1 implies that $x$ cannot be nilpotent; for $x$ was chosen in $N$ such that $\bar{\delta}_q(x) \notin N$ for some $q$. Thus some $r_i$ in $f(X)$ must be nonzero. We shall now show that this implies $\bar{\delta}_q(x) \in N$ for all $q \geqq 1$. Thus $N=0$ and therefore $(R')^-$ is a field. This implies that $p'$ is a maximal ideal and completes the proof of the theorem.

We shall show that $\bar{\delta}_q(x) \in N$ for all $q$ by induction on $q$.

Suppose $\bar{\delta}_1(x) \notin N$. We need the following lemma:

LEMMA 2. *If $\bar{\delta}_1(x) \notin N$, then $\bar{\delta}_k(x^l) \in N$ if $l>k$, and $\bar{\delta}_k(x^l) \notin N$ if $l=k$.*

PROOF. We prove this lemma via induction on $k$. If $k=1$, the result follows easily from the hypothesis and Leibnitz's rule. So assume the lemma holds if $k=1, \cdots, m$. Then

$$\bar{\delta}_{m+1}(x^{m+1}) = \sum_{i+j=m+1} \bar{\delta}_i(x)\bar{\delta}_j(x^m)$$

which in turn is congruent modulo $N$ to $\bar{\delta}_1(x)\bar{\delta}_m(x^m)$. But $\bar{\delta}_1(x)\bar{\delta}_m(x^m) \notin N$, therefore $\bar{\delta}_{m+1}(x^{m+1}) \notin N$.

If $l>1$, then $\bar{\delta}_{m+1}(x^{m+l}) \in N$ follows from repeated applications of Leibnitz's rule. □

Using Lemma 2, we can now show $\bar{\delta}_1(x) \notin N$ implies every $r_i$, $i=1, \cdots, n-1$, in $f(X)$ is zero. Applying $\bar{\delta}_1$ to the equation $f(x)=0$, we get

$$0 = \bar{\delta}_1(f(x)) = f'(x)\bar{\delta}_1(x) + \bar{\delta}_1(r_{n-1})x^{n-1} + \cdots + \bar{\delta}_1(r_1)x$$

where $f'(X)$ is the formal derivative of $f(X)$. Since $\bar{\delta}_1(x) \notin N$, $f'(x) \in N$. Therefore $r_1=0$. So

$$f(X) = X^n + r_{n-1}X^{n-1} + \cdots + r_2X^2.$$

If we apply $\bar{\delta}_2$ to $f(x)=0$, we get

$$0 = \bar{\delta}_2(x^n) + \sum_{i+j=2} \bar{\delta}_i(r_{n-1})\bar{\delta}_j(x^{n-1}) + \cdots + \sum_{i+j=2} \bar{\delta}_i(r_2)\bar{\delta}_j(x^2).$$

Applying Lemma 2, we get $r_2\bar{\delta}_2(x^2) \in N$. But $\bar{\delta}_2(x^2) \notin N$. Thus $r_2=0$. By applying $\bar{\delta}_3, \cdots, \bar{\delta}_{n-1}$ to $f(x)=0$ and applying Lemma 2, we get $r_{n-1}=\cdots=r_3=0$. This is a contradiction, and hence $\bar{\delta}_1(x) \in N$. So assume we have shown that $\bar{\delta}_1(x), \cdots, \bar{\delta}_{m-1}(x) \in N$ $(m>1)$. Assume $\bar{\delta}_m(x) \notin N$. Again we shall show that this leads to $r_{n-1}=\cdots=r_1=0$. We need:

LEMMA 3. *$\bar{\delta}_{km}(x^l) \in N$ if $l>k$, and $\bar{\delta}_{km}(x^l) \notin N$ if $l=k$.*

PROOF. We prove this lemma via induction on $k$. If $k=1$, then $\bar{\delta}_m(x) \notin N$ by hypothesis. $\bar{\delta}_m(x^2)=\sum_{i+j=m} \bar{\delta}_i(x)\bar{\delta}_j(x) \in N$ since $\bar{\delta}_1(x), \cdots, \bar{\delta}_{m-1}(x) \in N$. If $\bar{\delta}_m(x^l) \in N$ for $l=2, \cdots, t$, then $\bar{\delta}_m(x^{l+t}) \in N$ by an easy application of Leibnitz's rule. Hence the lemma holds for $k=1$.

Assume the result holds for $k=1, \cdots, r$. Applying Leibnitz's rule again, we get

$$\bar{\delta}_{(r+1)m}(x^{r+1}) = \sum_{i_1+\cdots+i_{r+1}=(r+1)m} \bar{\delta}_{i_1}(x) \cdots \bar{\delta}_{i_{r+1}}(x).$$

This sum is clearly congruent modulo $N$ to $\bar{\delta}_m(x)^{r+1}$ which is not an element of $N$. Thus $\bar{\delta}_{(r+1)m}(x^{r+1}) \notin N$. A similar argument shows

$$\bar{\delta}_{(r+1)m}(x^{r+l}) \in N \quad \text{if } l>1. \quad \square$$

If we now apply $\bar{\delta}_m$ to $f(x)=0$, we get

$$0 = \bar{\delta}_m(f(x)) = \bar{\delta}_m(x^n) + \sum_{i+j=m} \bar{\delta}_i(r_{n-1})\bar{\delta}_j(x^{n-1}) + \cdots + \sum_{i+j=m} \bar{\delta}_i(r_1)\bar{\delta}_j(x).$$

Applying Lemma 3 and the induction hypothesis, we get $r_1\bar{\delta}_m(x) \in N$. Hence $r_1=0$. Successively applying $\bar{\delta}_{2m}, \cdots, \bar{\delta}_{(n-1)m}$ to $f(x)=0$, we get $r_2=\cdots=r_{n-1}=0$. Thus every $r_i$ in $f(X)$ is zero. This is a contradiction. Hence $\bar{\delta}_q(x) \in N$ for all $q$. Hence the theorem is completely proven. $\square$

COROLLARY 1. *Let $R'$ be an integral extension of $R$, and let $R$ contain a field of characteristic zero. Let $\delta$ be an ordinary derivation of rank one on $R'$ such that $\delta$ restricts to a derivation on $R$. If $p$ is a $\delta$-differential prime ideal of $R$, then there exists a $\delta$-differential prime ideal $p'$ in $R'$ such that $p' \cap R=p$.*

PROOF. The corollary follows immediately from Theorem 1 since $\delta$ can be embedded as the first term in the higher derivation $\delta'=\{\delta^i/i!\}$ on $R'$. $\square$

We note that Corollary 1 is false in the characteristic $r\neq 0$ case. Consider the following simple example:

Let $Z_5$ denote the integers modulo 5. Let $X$ denote an indeterminate over $Z_5$ and consider $R'=Z_5[X]/(X^5)=Z_5[x]$. Then $R'$ is a local ring with maximal ideal generated by $x$. Clearly $R'$ is an integral extension of $Z_5$. Let $\delta$ be a derivation of rank one on $Z_5[X]$ defined by $\delta(X)=1$. Then $\delta(X^5)\subset (X^5)$. Hence, $\delta$ induces a derivation $\bar{\delta}: R'\to R'$ which restricts to a derivation on $Z_5$. Now (0) is a $\bar{\delta}$-differential prime ideal in $Z_5$, but there is no $\bar{\delta}$-differential prime ideal in $R'$ which lies over it. This example also shows us that the associated prime ideals of a differential ideal need not in general be differential. (This result is true in the characteristic zero case [3, Theorem 1].)

Using Theorem 1 we can easily prove the going up theorem for differential prime ideals.

COROLLARY 2. *With the same hypotheses as in Theorem* 1, *suppose* $p_1 \subset p_2 \subset \cdots \subset p_n$ *is a chain of* $\delta$*-differential prime ideals in* $R$. *Then there exists a chain of* $\delta$*-differential prime ideals* $p'_1 \subset \cdots \subset p'_n$ *in* $R'$ *such that* $p'_i \cap R = p_i$.

PROOF. By Theorem 1, there exists a $\delta$-differential prime ideal $p'_1$ in $R'$ lying over $p_1$. Passing to the residue class rings $R'/p'_1$ and $R/p_1$ and applying Theorem 1 again, we get a $\bar{\delta}$-differential prime ideal $q$ in $R'/p'_1$ which lies over $p_2/p_1$. We can now pull $q$ back to a $\delta$-differential prime ideal $p'_2$ which contains $p'_1$ and lies over $p_2$. Continuing in this fashion, we construct the entire chain. □

If we assume $R'$ is Noetherian, we can prove the going down theorem also.

THEOREM 2. *Suppose* $R'$ *is a Noetherian integral extension of* $R$. *Assume* $R$ *is a normal ring in which no nonzero element is a zero divisor in* $R'$. *Let* $\delta$ *be a higher derivation on* $R'$ *which restricts to a higher derivation on* $R$. *Let* $p \subset q$ *be a chain of* $\delta$*-differential prime ideals in* $R$, *and let* $q'$ *be a* $\delta$*-differential prime ideal in* $R'$ *lying over* $q$. *Then there exists a* $\delta$*-differential prime ideal* $p' \subset q'$ *such that* $p' \cap R = p$.

PROOF. Consider the $\delta$-differential ideal $pR'$. It is well known [**4**, p. 263] that any isolated prime of $pR'$ contracts to $p$ in $R$. Further, by [**2**, Proposition 1] or for a detailed proof [**1**, Theorem 1], any isolated prime of $pR'$ is $\delta$-differential. Since $pR' \subset q'$, $q'$ must contain some isolated prime $p'$ of $pR'$. This is the required $\delta$-differential prime ideal lying over $p$. □

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