PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

ORDERS IN SELF-INJECTIVE COGENERATOR RINGS

ROBERT C. SHOCK

ABSTRACT. This note states necessary and sufficient conditions for a ring to be a right order in certain self-injective rings. A ring R is said to have the dense extension property if each R-homomorphism from a right ideal of R into R can be lifted to some dense right ideal of R. A right ideal K is rationally closed if for each $x \in R-K$ the set $x^{-1}K=\{y \in R : xy \in K\}$ is not a dense right ideal of R. We state a major result. Let dim R denote the Goldie dimension of a ring R and $Z(R)$ the right singular ideal of R. Then R is a right order in a self-injective cogenerator ring if and only if R has the dense extension property, $Z(R)$ is rationally closed and the factor ring $R/Z(R)$ is semiprime with $\dim R/Z(R)=\dim R<\infty$.

1. **Introduction.** Let R always denote a ring with 1. Recall, a ring is quasi-Frobenius if R_R is right Artinian and R_R is injective. For other equivalent definitions see **[13]** and for the original definition see **[10]**. Barbara Osofsky **[13]** has studied a generalization of the quasi-Frobenius ring, the self-injective cogenerator ring. A ring R is a *self-injective cogenerator ring* if R_R is injective and R_R is a cogenerator in the category of unital right modules; this means that each right unital R-module can be embedded in a direct product of copies of R. The introduction of **[13]** gives a motivation for the study of these rings.

If R is a subring of Q and the identity of R is also the identity of Q then R is a *right order* in Q if

(a) every nonzero divisor of R is a unit in Q, and

(b) every element of Q can be written in the form of cd^{-1} where c and d are in R and d is a nonzero divisor of R.

The problem of finding suitable conditions for a ring R to be a right order in a quasi-Frobenius ring was first studied by Jans [7], then by Mewborn and Winton **[12]**, Shock (**[14]**, **[15]**), and Tachikawa **[16]**. These conditions require that R be a solid Goldie ring [7], that is, R satisfies the maximum condition on right annihilators of the injective hull of R. Proposition 8 replaces this solid Goldie condition by rings which satisfy

Received by the editors December 10, 1971.

AMS 1969 *subject classifications*. Primary 1625, 1650.

Key words and phrases. Injective, order, cogenerator, rationally closed, quasi-Frobenius, dense right ideal, complete ring of right quotients.

the minimum condition on annihilators of the injective hull of R. Our main result is an order theorem for self-injective cogenerator rings. The method of proof is via Theorem 5.3 of [**15**].

A right ideal U of R is called uniform if the intersection of any two nonzero right ideals of R contained in U is nonzero. If for some positive integer n the ring R contains an essential direct sum of n uniform right ideals of R then R is called a *right finite dimensional ring* and we write dim $R=n$. We denote the *right singular ideal* of R by $Z(R)$, the *complete ring of right quotients of R by Q* [**9**, p. 94] and the injective hull of R_R by $I(R)$. For $i \in I(R)$ and for a right ideal H of R we equate $i^{-1}H=\{x \in R : ix \in H\}$. A right ideal L of R is called *dense* if for $x \in R-0$ and $y \in R$ there is $r \in R$ such that $xr \neq 0$ and $yr \in L$; equivalently if $i \in I(R)$ and $iL=0$ then $i=0$ [**9**, p. 96]. Shock showed that a right ideal K of R is an annihilator of a subset of $I(R)$ if and only if K is a rationally closed right ideal of R [**15**]. For a general reference see [**9**].

In proving the main result which was stated in the abstract we first prove that a self-injective ring is a cogenerator if and only if the ring is right finite dimensional and the right singular ideal is rationally closed (Proposition 2). Finally, a ring R is a right order in a quasi-Frobenius ring if and only if R has the dense extension property and R satisfies the minimum condition on rationally closed right ideals of R and $Z(R)$ is the prime radical of R.

2. **Self-injective cogenerator rings.** We list a series of propositions which lead to our main theorem.

PROPOSITION 1 (OSOFSKY [**13**]). *A self-injective cogenerator ring is a right finite dimensional ring.*

PROOF. See Theorem 1 of [**13**].

It is clear that if R contains no proper dense right ideals then every right ideal is rationally closed and conversely.

PROPOSITION 2. *Let R be a self-injective ring. Then R is a cogenerator if and only if R is right finite dimensional and Z(R) is rationally closed.*

PROOF. Since R_R is injective, R is a cogenerator if and only if each simple module is embedded in R [**13**, Lemma 1]. Equivalently, if M is a maximal right ideal of R then $iM=0$ for some $i \in R-(0)$ which means that R has no proper dense right ideals. If R is a cogenerator then every right ideal is rationally closed and R is finite dimensional by Proposition 1. For the converse suppose that D is a dense right ideal of R. Since $Z(R)$ is rationally closed, $(D+Z(R))/Z(R)$ is dense in $R/Z(R)$ [**15**, Proposition 5.1]. However, $R/Z(R)$ is a semiprime Artinian ring because R is a finite dimensional ring [9, p. 103]. Therefore, $R/Z(R)$ contains no proper dense right ideals and thus $D+Z(R)=R$. Since $Z(R)$ is the Jacobson radical of R and is therefore a small submodule we have $D=R$. This completes the proof.

PROPOSITION 3. *The complete ring of right quotients Q of R is a self-injective ring if and only if R has the dense extension property.*

PROOF. This is an immediate consequence of the Findlay-Lambek-Gabriel characterization of the complete ring of quotients as in [3], [4], or [6].

The most general condition for a ring to have a right classical quotient ring is the Ore condition [9, p. 109]. In working with theorems on orders there is always a method which in some sense replaces the Ore condition. We now state our method.

PROPOSITION 4 (SHOCK [15]). *Suppose that $Z(Q)$ is the Jacobson radical of Q and is rationally closed. If $Q/Z(Q)$ is a semiprime Artinian ring and $R/Z(R)$ is semiprime then R is a right order in Q.*

PROOF. We sketch a proof of Theorem 5.3 of [15]. Let $R/Z(Q)$ denote the embedded image of $R/Z(R)$ in $Q/Z(Q)$. Let D be a dense right ideal of R. Since $Z(R)$ is rationally closed $(D+Z(Q))/Z(Q)$ is dense in $Q/Z(Q)$. Hence $Q/Z(Q)$ is the complete ring of right quotients of $R/Z(Q)$ and $(D+Z(Q))/Z(Q)$ contains an invertible element $d+Z(Q)$ in $Q/Z(Q)$. Also d is invertible in R since $Z(Q)$ is the Jacobson radical. A nonzero divisor b of R is a right nonzero divisor of $R/Z(Q)$ and $b+Z(Q)$ is invertible in $Q/Z(Q)$ and as before b is invertible in R.

Recall Q denotes the complete ring of right quotients of R. It is well known and easy to see that dim $R=$dim Q whenever R or Q is a right finite dimensional ring.

THEOREM 5. *A ring R is a right order in a self-injective cogenerator ring if and only if*

(I) *R has the dense extension property, and*

(II) *the right singular ideal $Z(R)$ is rationally closed and the factor ring $R/Z(R)$ is semiprime with* dim $R/Z(R)=$dim $R<\infty$.

PROOF. Assume that R is a right order in Q and Q is a self-injective cogenerator ring. Then Q has no proper dense right ideals and $Z(Q)$ is rationally closed. Since $Z(Q)\cap R=Z(R)$, $Z(R)$ is rationally closed. Since Q is a classical ring of right quotients of R it follows that $Q/Z(Q)$ is semiprime Artinian and is a classical ring of right quotients of $R/Z(R)$. Hence $R/Z(R)$ is semiprime and dim $R/Z(R)=$dim $Q/Z(Q)$. Also, dim $R=$dim Q and finally dim $Q=$dim $Q/Z(Q)$ because idempotents may be lifted modulo

$Z(Q)$ [**9**, p. 103] and primitive idempotents remain primitive [**9**, p. 75]. Therefore, dim R=dim $R/Z(R)$. For the converse let Q denote the complete ring of right quotients of R. By (I) and by Proposition 3 the ring Q is self-injective. Also, dim Q=dim R and Q is semiperfect [**9**, p. 103]. Therefore, dim Q=dim $Q/Z(Q)$ because primitive idempotents remain primitive modulo $Z(Q)$. If $Z(Q)$ is rationally closed then by Proposition 4 the ring R is a right order in Q. Furthermore the proof of Theorem 5.3 of [**15**] (or the proof of Proposition 2) shows that Q has no proper dense right ideals. Thus, Q is a self-injective cogenerator ring. It suffices to show that $Z(Q)=H$ where $H=r(l(Z(Q)))$ and $l(Z(Q))=\{x \in Q : xZ(Q)=0\}$ and $r(l(Z(Q)))=\{y \in Q : l(Z(Q))y=0\}$. Clearly, $H \supseteq Z(Q)$ and if $x \in Q$ then $xZ(R)=0$ if and only if $xZ(Q)=0$. Also, H is an ideal since $Z(Q)$ is. Hence, $H=r(l(Z(R)))$. If $y \in R-Z(R)$ there is $i \in I(R)=Q$ such that $iZ(R)=0$ and $iy \neq 0$. We conclude that $y \in R-H$ and $H \cap R=Z(R)$. This means that $R/Z(R)$ is embedded in Q/H and by Lemma 5.2 of [**15**], Q/H is a ring of right quotients of the embedded ring of $R/Z(R)$. Therefore, dim Q/H= dim $R/Z(R)$. Recall dim $Q/Z(Q)$=dim Q=dim R. By (II), dim R= dim $R/Z(R)$ and we have dim Q/H=dim $Q/Z(Q)$. Therefore, $H=Z(Q)$ because $Q/Z(Q)$ is a semiprime Artinian ring and $H \supseteq Z(Q)$.

An ideal L of R is left T-nilpotent if for each sequence $x_1, x_2, \cdots$ of L there is a corresponding integer n such that $x_1 x_2 \cdots x_n=0$. Assume that R modulo the Jacobson radical J of R is a semiprime Artinian ring. Then R is *left perfect* if J is left T-nilpotent. Also R is a *semiprimary ring* if J is nilpotent.

Corollary 6. *A ring R is a right order in a self-injective cogenerator ring which is left perfect* (*semiprimary*) *if and only if the prime radical of R is left T-nilpotent* (*nilpotent*) *and* (I) *and* (II) *of Theorem* 5 *hold.*

Proof. Suppose R is a right order in Q. If $a_i b_i^{-1} \in Z(Q)$ for $1 \leqq i \leqq n$ then $a_1 b_1^{-1} \cdots a_n b_n^{-1} = a_1 x_2 \cdots x_n a^{-1}$ where $x_i \in Z(R)$ and $b_i^{-1} a_{i+1} = x_{i+1} c_i^{-1}$ for appropriate $c_i \in R$, $1 \leqq i \leqq n-1$. The proof is straightforward and details are omitted.

Lemma 7. *If R has the minimum condition on rationally closed right ideals then so does Q and conversely.*

Proof. Suppose K and H are two rationally closed right ideals of Q with $K \supsetneqq H$. There is $i \in I(R)$ such that $iH=0$ and $ik \neq 0$ for some $k \in K$. Since $k \in Q$, $k^{-1}R$ is dense [**9**, p. 96]. Thus, $(ik)(k^{-1}R) \neq 0$ and $i(K \cap R) \neq 0$ implies that $(K \cap R) \supsetneqq (H \cap R)$. Also $K \cap R$ and $H \cap R$ are rationally closed in R and the result follows.

PROPOSITION 8. *A ring R is a right order in a quasi-Frobenius ring if and only if*

(1) *R has the dense extension property,*

(2) *R satisfies the minimum condition on rationally closed right ideals of R, and*

(3) *the prime radical of R is the right singular ideal of R.*

PROOF. Suppose R is a right order in a quasi-Frobenius ring Q. Proposition 3 implies (1) and Lemma 7 implies (2). Since R is a right order in Q, $R/Z(R)$ is a right order in $Q/Z(Q)$ and hence $R/Z(R)$ is semiprime. Also $Z(R)$ is nilpotent since $Z(Q)$ is and (3) follows. For the converse, (1) and (2) imply that Q is a self-injective ring with the minimum condition on right annihilators. This implies the maximum condition on left annihilators of the ring Q. Since Q_Q is injective, $l(r(L))=L$ where $r(L)=\{x \in Q : Lx=0\}$, $l(r(L))=\{y \in Q : y(r(L))=0\}$ for every finitely generated left ideal L of Q. The collection of finitely generated left ideals of Q satisfies the maximum condition and ${}_QQ$ is left Noetherian. Therefore, Q is quasi-Frobenius and Q with (3) above satisfies the hypothesis of Proposition 4. Therefore, R is a right order in Q.

ADDED IN PROOF. Morita [Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142] introduced injective cogenerator rings and showed that the Artinian ones were quasi-Frobenius. Walker and Faith [J. Algebra **5** (1967), 203–221] introduced cogenerator rings and showed that the semilocal ones were self-injective.

REFERENCES

1. H. Bass, *Finistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488. MR **28** #1212.

2. C. Faith, *Lectures on injective modules and quotient rings*, Lecture Notes in Math., no. 49, Springer-Verlag, Berlin and New York, 1967. MR **37** #2791.

3. G. D. Findlay and J. Lambek, *A generalized ring of quotients*. I, II, Canad. Math. Bull. **1** (1958), 77–85; 155–167. MR **20** #888.

4. G. Gabriel, *Des catégories abeliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

5. A. W. Goldie, *Semi-prime rings with the maximum condition*, Proc. London Math. Soc. (3) **10** (1960), 201–220. MR **22** #2627.

6. O. Goldman, *Rings and modules of quotients*, J. Algebra **13** (1969), 10–47. MR **39** #6914.

7. J. P. Jans, *On orders in quasi-Frobenius rings*, J. Algebra **7** (1967), 35–43. MR **35** #2924.

8. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. **2** (1951), 891–895. MR **13**, 618.

9. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass. 1966. MR **34** #5857.

10. T. Nakayama, *On Frobeniusean algebras*. I, Ann. of Math. (2) **40** (1939), 611–633 MR **1**, 3.

11. T. Nakayama, *On Frobeniusean algebras*. II, Ann. of Math. (2) **42** (1941), 1–21. MR **2**, 344.

12. A. C. Mewborn and C. N. Winton, *Orders in self-injective semi-perfect rings*, J. Algebra **13** (1969), 5–9. MR **39** #6925.

13. B. L. Osofsky, *A generalization of quasi-Frobenius rings*, J. Algebra **4** (1966), 373–387; Erratum, ibid. **9** (1968), 120. MR **34** #4305; MR **36** #6443.

14. R. C. Shock, *Right orders in self-injective rings*, Notices Amer. Math. Soc. **17** (1970), 561. Abstract #70T–A76.

15. ———, *Injectivity, annihilators, and orders*, J. Algebra **19** (1971), 96–103.

16. H. Tachikawa, *Localization and Artinian quotient rings*, Math. Z. **119** (1971), 239–253.

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901