A CHARACTERIZATION OF EXCHANGE RINGS

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ABSTRACT. A necessary and sufficient condition on the endomorphism ring of a module for the module to have the finite exchange property is given. This condition is shown to be strictly weaker than a sufficient condition given by Warfield. The class of rings having these properties is equationally definable and is a natural generalization of the class of regular rings. Finally, it is observed that in the commutative case the category of such rings is equivalent with the category of ringed spaces $(X, \mathscr{R})$ with X a Boolean space and $\mathscr{R}$ a sheaf of commutative (not necessarily Noetherian) local rings.

In all that follows we will consider modules and morphisms as being over a ring S. Given two direct decompositions $K = M \oplus X = \oplus \sum \{A_i | i \in I\}$ of a module, we say that they can be exchanged at M if there are submodules $A_i' \subseteq A_i$ $(i \in I)$ such that $K = M \oplus (\oplus \sum \{A_i' | i \in I\})$. A module M is said to have the n-exchange property if any pair of decompositions $K = M' \oplus X = \oplus \sum \{A_i | i \in I\}$ with $M' \cong M$ and $\mathrm{card}(I) \leq n$ can be exchanged at M'. The module M has the (finite) exchange property if it has the n-exchange property for all (finite) cardinals n. In a fundamental paper [1], Crawley and Jónsson define the exchange properties and use them to prove theorems on isomorphic refinements of direct decompositions of modules. Warfield has shown [3] that an indecomposable module has the exchange property if and only if its endomorphism ring is local. In an attempt to generalize this theorem he has shown recently [4] that if R is $\mathrm{End}_S(M)$ and J is its Jacobson radical, then M has the finite exchange property if every principal left ideal of R/J is of the form $Re + J$ where e is an idempotent element in R. In the present paper we will give another condition on R which is, in fact, necessary and sufficient for M to have the finite exchange property and furthermore is strictly weaker than Warfield's.

We will frequently use the fact that a direct decomposition $K = A \oplus B$ of a module is naturally related to an idempotent endomorphism π of K which leaves the elements of A fixed and has B as its kernel. This will be

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referred to as the projection on A along B and denote 1 by $K=A \oplus B$; $\pi: K \twoheadrightarrow A$. We will routinely abuse notation by beginning with a morphism $f: A_1 \rightarrow A_2$ and submodules $B_i \subseteq A_i$ ($i=1, 2$) such that $f(B_1) \subseteq B_2$ and writing $f: B_1 \rightarrow B_2$ instead of $(f | B_1): B_1 \rightarrow B_2$.

Our point of view is that if $K=M \oplus X=A \oplus B$ and π is the projection on A along B, then exchange at M can be characterized in terms of the action of π on M, which is, in turn, describable as a first order sentence about $R=\text{End}_S(M)$.

DEFINITION. Given a module K with submodule M, the morphism α of M into K is *divided* on M if $M=M_1 \oplus M_2$ and $K=K_1 \oplus K_2$ such that (i) $\alpha: M_1 \cong K_1$, (ii) $\alpha(M_2) \subseteq K_2$, (iii) $1-\alpha: M_2 \twoheadrightarrow K$ splits.

It is not difficult to show that the endomorphism ring R of a module M is regular if and only if given $\alpha \in R$, $M=M_1 \oplus \ker(\alpha)$ such that $\alpha: M_1 \twoheadrightarrow M$ splits. This condition clearly implies that $\alpha: M \rightarrow M$ is divided on M.

LEMMA 1. *Given a module K with submodule M, the morphism $\alpha: M \rightarrow K$ is divided on M if and only if there are morphisms γ and σ in $\text{Hom}_S(K, M)$ such that $\gamma \alpha \gamma = \gamma$ on K, and $\sigma(1_M-\alpha)(1_M-\gamma\alpha)=1_M-\gamma\alpha$ on M.*

PROOF. Given α, γ and σ with these properties, observe that $\alpha\gamma$ and $\gamma\alpha$ are idempotent endomorphisms of K and M respectively and that $(1_M-\alpha):(1_M-\gamma\alpha)(M) \twoheadrightarrow K$ splits. But then α divides on M because we have the decompositions

$$M = M_1 \oplus M_2; \quad \gamma\alpha(M) = M_1, \qquad K = K_1 \oplus K_2; \quad \alpha\gamma(K) = K_1.$$

Conversely, if $M=M_1 \oplus M_2$ and $K=K_1 \oplus K_2$ such that $\alpha: M_1 \cong K_1$, $\alpha(M_2) \subseteq K_2$ and $(1_M-\alpha): M_2 \twoheadrightarrow K$ splits, then, defining γ on K_1 as $(\alpha | M_1)^{-1}$ and on K_2 as 0, we have that $\alpha\gamma$ is the projection on K_1 along K_2, while $\gamma\alpha$ is the projection on M_1 along M_2. But then the equations follow easily.

LEMMA 2. *The decompositions $K=M \oplus X$, $K=A \oplus B$, $\pi(K)=A$ can be exchanged at M if and only if π is divided on M.*

PROOF. If $K=M \oplus A_1 \oplus B_1$ with $A_1 \subseteq A$ and $B_1 \subseteq B$, then, letting

$$M_1 = M \cap (A + B_1), \qquad M_2 = M \cap (A_1 + B),$$

$$A_2 = A \cap (M + B_1), \qquad K_2 = A_1 + B,$$

it can be shown easily that $M=M_1 \oplus M_2$, $K=A_2 \oplus K_2$, $\pi: M_1 \cong A_2$, and $\pi(M_2) \subseteq K_2$. Since $(1_M-\pi)$ is monic on M_2 and $(1-\pi)(M_2)=B \cap (M+A_1)$, which is a complement of $A+B_1$ in K, we conclude that π is divided on M.

For the converse suppose that there are endomorphisms γ and σ such that $\gamma\pi\gamma=\gamma$ on K and

$$(1) \qquad\qquad \sigma(1_M - \pi)(1_M - \gamma\pi) = 1_M - \gamma\pi \quad \text{on } M.$$

Then, noting that if ρ is the projection on M along X, $\gamma = \rho\gamma$, $(1_M - \gamma\pi)\rho = \rho(1_M - \gamma\pi)\rho = (1_K - \gamma\pi)\rho$ and $(1_M - \pi)\rho = (1_K - \pi)\rho$, multiplying (1) on the right by ρ and on the left by $(1_K - \gamma\pi)$, we obtain

$$(2) \qquad (1_K - \gamma\pi)\sigma(1_K - \pi)(1_K - \gamma\pi)\rho = (1_K - \gamma\pi)\rho \quad \text{on } K.$$

Now if we let θ_1 be the idempotent $\gamma\pi$ and $\theta_2 = (1_K - \theta_1)\sigma(1_K - \pi)(1_K - \theta_1)$, we infer from (2) that θ_2 is an idempotent such that

$$\theta_1\rho + \theta_2\rho = \rho, \qquad \theta_1\theta_2 = \theta_2\theta_1 = 0$$

and

$$\theta_2\pi(1_K - \theta_1) = (1_K - \theta_1)\sigma(1_K - \pi)(\pi - \theta_1)(1_K - \theta_1) = 0.$$

Thus, if we let $\theta_i(K) = M_i$, $\ker \theta_i = L_i$ $(i = 1, 2)$, we obtain $M = M_1 + M_2$:

$$K = M_1 \oplus L_1, \qquad B, M_2 \subseteq L_1,$$
$$K = M_2 \oplus L_2, \qquad M_1, (A \cap L_1) \subseteq L_2,$$

so that

$$L_1 = (A \oplus B) \cap L_1 = (A \cap L_1) \oplus B,$$
$$L_1 \cap L_2 = (A \cap L_1) \oplus (B \cap L_2),$$
$$L_2 = (M_1 \oplus L_1) \cap L_2 = M_1 \oplus (L_1 \cap L_2)$$
$$= M_1 \oplus (A \cap L_1) \oplus (B \cap L_2)$$

and

$$K = M_2 \oplus M_1 \oplus (A \cap L_1) \oplus (B \cap L_2)$$
$$= M \oplus (A \cap L_1) \oplus (B \cap L_2),$$

which is the desired result.

THEOREM 1. *The module M has the finite exchange property if and only if the ring $R = \mathrm{End}_S(M)$ has the property*

(E) *Given $\alpha \in R$, there exist γ and σ in R such that $\gamma\alpha\gamma = \gamma$ and $\sigma(1 - \alpha)(1 - \gamma\alpha) = 1 - \gamma\alpha$.*

PROOF. We first recall that, according to Crawley and Jónsson [1, Lemma 3.11], the finite exchange property for a module M is equivalent to the 2-exchange property. If the condition (E) holds for R and

$$(3) \qquad K = M \oplus X = A \oplus B, \qquad \rho(K) = M, \qquad \pi(K) = A,$$

then $\rho\pi\rho$, viewed as an endomorphism of M, is divided on M. It is then easily seen that π as a morphism of M into K is divided on M, whence the decompositions (3) can be exchanged at M and M has the finite exchange property.

To prove the converse we fix $\alpha \in R$ and define $K = M \times M$,

$$M \underset{\tau_1}{\overset{\varepsilon_1}{\rightleftharpoons}} K \underset{\tau_2}{\overset{\varepsilon_2}{\leftrightharpoons}} M$$

$\sigma(m) = (\alpha(m), (1-\alpha)(m))$ and $\Delta(m) = (m, -m)$. Then let M_σ, M_Δ, A and B be the images of M in K under σ, Δ, ε_1 and ε_2 respectively. It is clear that σ is an isomorphism of M onto M_σ and that $K = M_\sigma \oplus M_\Delta = A \oplus B$ with $\varepsilon_1 \tau_1$ the projection on A along B. Since M has the 2-exchange property, these decompositions can be exchanged at M_σ and $\varepsilon_1 \tau_1$ is divided on M_σ with $M_\sigma = M_\sigma^{(1)} \oplus M_\sigma^{(2)}$ and $K = A_2 \oplus K_2$ the related decompositions. Inasmuch as $A_2 \subseteq A$, we infer from the second decomposition that $A = A_2 \oplus (A \cap K_2)$. Now the isomorphisms $\sigma: M \cong M_\sigma$ and $\varepsilon_1: M \cong A$ give decompositions $M = M_1 \oplus M_2 = M_3 \oplus M_4$ where $\sigma(M_i) = M_\sigma^{(i)}$ $(i=1, 2)$, $\varepsilon_1(M_3) = A_2$ and $\varepsilon_1(M_4) = A \cap K_2$. It is then easy to show that $\tau_1(\varepsilon_1\tau_1)\sigma: M_1 \cong M_3$, $\tau_1(\varepsilon_1\tau_1)\sigma(M_2) \subseteq M_4$ and that $\tau_2(1-\varepsilon_1\tau_1)\sigma: M_2 \rightarrowtail M$ splits. But since $\alpha = \tau_1(\varepsilon_1\tau_1)\sigma$ and $(1-\alpha) = \tau_2(1-\varepsilon_1\tau_1)\sigma$, we conclude that α is divided on M which gives the desired result.

Warfield defines a ring R to be an exchange ring if $_R R$ has the exchange property (as an R-module), but then goes on to show that such rings are exactly the rings of endomorphisms of modules with the finite exchange property [4, Theorem 2]. Thus we have characterized exchange rings in Theorem 1. As was mentioned in the introduction, Warfield also proves [4, Theorem 3] that if R is a ring such that every principal left ideal of R/J is of the form $Re + J$ where $e = e^2$ and J is the Jacobson radical of R, then R is an exchange ring. Our next theorem shows that this does not characterize exchange rings.

THEOREM 2. *There is a commutative exchange ring which is Jacobson semisimple but not regular.*

PROOF. Let $\hat{Z}_p$ be the p-adic integers, F any field having $\hat{Z}_p$ as a subring and A the ring product of $\aleph_0$ copies of F. Viewing A as sequences of elements from F, let R be the subring of A generated by the finitely nonzero sequences of A (called A_0) and multiples from $\hat{Z}_p$ of $(1, 1, 1, \cdots)$. It is clear that if, for $a \in A$, we denote by $\pi_i(a)$ the ith coordinate of a, then $\pi_i: R \rightarrow F$ so that $\ker \pi_i$ is a maximal ideal in S. Inasmuch as $\bigcap \{\ker \pi_i | i = 1, 2, 3, \cdots\} = (0)$, R is a Jacobson semisimple ring. But R cannot be regular because the factor rings of such a ring would be regular and $\hat{Z}_p$, which is isomorphic to $R/(A_0)$, is not regular. To see that R is an exchange ring, begin with a typical element $a = (a_1, a_2, \cdots, a_m, r, r, \cdots)$ in R where $a_i \in F$, $1 \le i \le m$, and $r \in \hat{Z}_p$. For each $i = 1, 2, \cdots, m$ such that $a_i \neq 0$, let $c_i = a_i^{-1}$ and $d_i = 0$. For $i = 1, 2, \cdots, m$ such that $a_i = 0$ let $c_i = d_i = 0$. Then for $1 \le i \le m$, $c_i a_i c_i = c_i$ and $d_i(1-a_i)(1-a_i c_i) = 1 - a_i c_i$.

If $r \notin p\ \hat{Z}_p$, let $s = r^{-1}$ in $\hat{Z}_p$ and $t = 0$. If $r \in p\hat{Z}_p$, let $s = 0$ and $t = (1 - r)^{-1}$. In either case $srs = s$ and $t(1 - r)(1 - rs) = 1 - rs$, and if we let $c = (c_1, \cdots, c_m, s, s, \cdots)$ and $d = (d_1, \cdots, d_m, t, t, \cdots)$ the desired equations hold for a, c and d.

Constructions of the sort used in the preceding proof can be viewed as a special case of the formation of the ring of sections of a sheaf of rings over a Boolean space. In view of the fact that Pierce [2] uses this technique to such advantage in his study of commutative regular rings, it seems natural to study commutative exchange rings in connection with ringed spaces, although this is far removed from the original source of exchange rings. However, recalling Warfield's characterization of an exchange ring with no nontrivial idempotents as a local ring (in the sense of having a unique maximal ideal) [2, Proposition 1], we have the following pleasant result.

THEOREM 3. *There is an equivalence between the category of commutative exchange rings and the category of ringed spaces $(X, \mathscr{R})$ with X a Boolean space and $\mathscr{R}$ a sheaf of commutative local rings.*

PROOF. First observe that if $(X, \mathscr{R})$ is a ringed space with X a Boolean space and $\mathscr{R}$ a sheaf of commutative local rings, then each of the stalks satisfies the property (E), so that, by Pierce's Theorem 3.4 [2], the ring of sections of $(X, \mathscr{R})$ does likewise. On the other hand, the stalks of the ringed space of a commutative exchange ring are commutative exchange rings (since the property (E) is preserved by epimorphisms) with no nontrivial idempotents and hence are local. Pierce's Theorem 10.1 then gives the equivalence of these categories.

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