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THE SCHUR INDEX AND ROOTS OF UNITY

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ABSTRACT. A short proof is given for the main step in the proof of the theorem of Benard and Schacher which asserts that if the Schur index of a character χ of a finite group is m then the mth roots of unity lie in the field of values $Q(\chi)$.

One of the most elegant results about the Schur index of a character of a finite group is the recent one by M. Benard and M. Schacher. They prove the Schur index over the rationals of the character χ can be m only when the field of values of χ contains the mth roots of unity. Their proof uses the Witt-Brauer reduction to the case of a "special" character, the theory of Hasse invariants and a rather long computational theorem of C. Ford **[1]**. We give here a proof of Ford's result (including a case not covered by him) in a version which covers the bulk of the Benard-Schacher result. The proof uses only the most elementary facts about crossed products and the Brauer group. In order to make the proof even shorter, we shall state it as a result about algebras. For the passage from characters to special characters and then to algebras, the reader may consult any of the references listed.

Let K be a field. A cyclotomic algebra over K is a crossed product $\Delta=(K(\varepsilon), G, \beta)=\sum_{\sigma\in G} K(\varepsilon)u_\sigma$ in which ε is a root of unity, G is the Galois group of $K(\varepsilon)$ over K, and β is a factor set whose values are roots of unity in $K(\varepsilon)$. The multiplication in Δ is given by the rules

$$u_\sigma x = \sigma(x)u_\sigma, \qquad u_\sigma u_\tau = \beta(\sigma, \tau)u_{\sigma\tau}$$

for all x in $K(\varepsilon)$ and σ, τ in G.

THEOREM. *If the cyclotomic algebra Δ has exponent m in the Brauer group of K, then the mth roots of unity lie in the center K.*

PROOF. We shall change factor sets in the proof so we write $\Delta(\beta)$ for the crossed product made with $K(\varepsilon)$, G and β. Let the values of β generate a group $\langle\varepsilon_n\rangle$ of nth roots of unity. In the Brauer group of K, we have

$$[\Delta(\beta)]^n = [\Delta(\beta^n)] = [\Delta(1)] = 1.$$

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By assumption the order of $[\Delta(\beta)]$ is m so m divides n. Thus a primitive mth root ε_m lies in $K(\varepsilon)$ and it is necessary to show it lies in K. We show this by proving ε_m is fixed by all elements in G. Let $\tau \in G$ and let $\Delta(\beta^\tau)$ denote the crossed product $(K(\varepsilon), G, \beta^\tau) = \sum_{\sigma\in G} K(\varepsilon)v_\sigma$, where the new factor set β^τ is obtained by applying τ to the factor set β. It is necessary to check that β^τ is indeed a factor set. This is not in general true but works here because G is abelian. The verification is left to the reader. Now map $\Delta(\beta)$ to $\Delta(\beta^\tau)$ by $\sum x_\sigma u_\sigma \to \sum \tau(x_\sigma)v_\sigma$. One may now check this is a K-algebra isomorphism and so $[\Delta(\beta)] = [\Delta(\beta^\tau)]$ in the Brauer group of K. Now there is a positive integer r such that $\tau(\varepsilon_n) = (\varepsilon_n)^r$. It follows that $\beta^\tau = \beta^r$. Thus

$$[\Delta(\beta)] = [\Delta(\beta^\tau)] = [\Delta(\beta)]^r.$$

Since $[\Delta(\beta)]$ has order m we find m divides $r-1$; or $r = 1+mk$. Now ε_m lies in $\langle\varepsilon_n\rangle$ so $\tau(\varepsilon_m) = (\varepsilon_m)^r = (\varepsilon_m)^{1+mk} = \varepsilon_m$ which proves ε_m is fixed by all elements of G as required.

References

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