ON THE EQUIVALENCE OF THREE
LOCAL THEOREM TECHNIQUES

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ABSTRACT. A result formulated by P. Hall which is used in
obtaining local theorems in group theory is shown to be equivalent
to the projection set theorem of Kuroš and to the existence of
Malcev vectors for local systems.

0. **Introduction.** It was Malcev [3] who first stated and proved local
theorems in group theory. To explain the term "local theorem" we need
some preliminary definitions.

Let S be a set. A local system L on S is a collection of subsets of S
such that for each finite subset $\{x_1, \cdots, x_n\} \subseteq S$ there is an $H \in L$ with
$\{x_1, \cdots, x_n\} \subseteq H$. It follows immediately if L_1 is a local system on S
and L_2 is a collection of subsets of S with $L_1 \subseteq L_2$ then L_2 is a local system
on S.

Let Σ be a class of groups. We say that a group $G \in L(\Sigma)$ if there is a
local system on G consisting of subgroups in Σ. We always have $\Sigma \subseteq L(\Sigma)$.
A local theorem for Σ asserts that $\Sigma = L(\Sigma)$. For examples of such classes
see Kuroš [2] or D. J. S. Robinson [6].

The original methods of Malcev involved the use of logical techniques.
However, the needed technical theorem can be stated without any refer-
ence to first order logic. One such formulation due to P. Hall appears with
a proof in D. J. S. Robinson [6]. Another version due to A. Robinson
and very similar to Hall's appears in [5]. We now state a variant of Hall's
theorem.

THEOREM H. *Let L be a local system on S, F a set, n a positive integer.
Suppose that for each $H \in L$ there is a function $f_H: H^n \rightarrow F$ and
$\{f_H(x) \mid H \in L\}$ is finite for each $x \in S^n$. Then there is a function $f: S^n \rightarrow F$
such that for any finite subset $K \subseteq S^n$ there is an $H \in L$ with $K \subseteq H^n$ and
$f \mid K = f \mid H$.*

REMARKS. In P. Hall's original formulation F is a finite set. One can derive Theorem H from Hall's theorem.

In §1 we apply Theorem H to inverse limits systems to obtain the projection set theorem of Kuroš. In §2 we use Kuroš's theorem to prove the existence of Malcev vectors which in turn yields a proof of Theorem H. We discuss the strength of these theorems in §3.

1. **Inverse limit systems.** Let $\langle I, \leq \rangle$ be an upper directed partially ordered set. A collection of sets $\{A_i | i \in I\}$ and functions $\{\varphi_{ij} | i, j \in I, i \leq j\}$ is called an inverse limit system if

(1) $\varphi_{ij} : A_j \rightarrow A_i$;
(2) φ_{ii} is the identity on A_i; and
(3) $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$, $i \leq j \leq k$. The inverse limit A^∞ of such a system is

$$\left\{ p \in \prod_{i \in I} A_i \,\middle|\, \forall i, j \,\exists k \; i, j \leq k \;\exists a \in A_k \; \varphi_{ik}(a) = p(i) \,\&\, \varphi_{jk}(a) = p(j) \right\}.$$

Using Theorem H we now prove a form of Kuroš's projection set theorem [**2**, p. 167].

THEOREM K.   *If $\{A_i\}$, $\{\varphi_{ij}\}$ is an inverse limit system with each $A_i \neq \varnothing$ and finite, then $A^\infty \neq \varnothing$.*

PROOF. Using the axiom of choice for families of finite sets form $\{a_i | i \in I, a_i \in A_i\}$. For each $j \in I$ let $H_j = \{i | i \leq j\}$. Let $L = \{H_j | j \in I\}$. L is a local system on I. For each $H_i \in L$ define $f_{H_i} : H_i \rightarrow \bigcup_{i \in I} A_i$ as follows: $f_{H_i}(j) = \varphi_{ji}(a_i)$. Since $\{f_{H_i}(j) | H_i \in L\} \subseteq A_j$ and A_j is finite, we can apply Theorem H and obtain a function $f : I \rightarrow \bigcup_{i \in I} A_i$. We now prove $f \in A^\infty$. Clearly $f(i) \in A_i$. Let $K = \{i, j\}$. There is an H_k, $K \subseteq H_k$ and $f_{H_k} | K = f | K$. Hence $f(i) = \varphi_{ik}(a_k)$ and $f(j) = \varphi_{jk}(a_k)$. Therefore $f \in A^\infty$.

2. **Finite partitions of a local system.** Let L be a local system on S. A finite partition σ of L is a set $\{L_i^\sigma | i \leq n\}$ such that $L_i^\sigma \cap L_j^\sigma = \varnothing$ for $i \neq j$ and $\bigcup \{L_i^\sigma | i \leq n\} = L$. The following is easily proven:

LEMMA 2.0.   *If L is a local system on S and $\{L_i^\sigma | i \leq n\}$ is a partition of L, then there is an $i_0 \leq n$ with $L_{i_0}^\sigma$ a local system on S.*

DEFINITION.   *L is a local system.* Let $P = \{\sigma | \sigma$ a finite partition of $L\}$. A Malcev vector for L is a function f on P such that $f(\sigma)$ is an element of the partition σ, and for any $\{\sigma_1, \cdots, \sigma_n\} \subseteq P$ we have $\bigcap \{f(\sigma_i) | i \leq n\}$ is a local system on S.

The following theorem on Malcev vectors appears in McLain [**4**]. Our proof is based on Theorem K.

THEOREM M.   *Let L be a local system on S. There is a Malcev vector for L.*

PROOF.   Let $P=\{\sigma|\sigma$ a finite partition of $L\}$. We partially order P: $\sigma\leqq\tau$ iff τ refines σ, i.e., each element of τ is contained in a (unique) element of σ. $\langle P, \leqq \rangle$ is upper directed since any σ_1, σ_2 have a common refinement τ. For each $\sigma \in P$ let $A_\sigma=\{L_i^\sigma|L_i^\sigma$ is a local system on $S\}$. A_σ is clearly finite and $A_\sigma\neq\varnothing$ by Lemma 2.0. For $\sigma\leqq\tau$ define $\varphi_{\sigma\tau}:A_\tau\to A_\sigma$ by $\varphi_{\sigma\tau}(L_i^\tau)=$ the unique L_j^σ such that $L_i^\tau\subseteq L_j^\sigma$. Clearly $\{A_\sigma|\sigma \in P\}$, $\{\varphi_{\sigma\tau}|\sigma\leqq\tau;\ \sigma, \tau \in P\}$ is an inverse limit system. By Theorem K there is an $f\in A^\infty, f:P\to \bigcup A_\sigma$, $f(\sigma) \in A_\sigma$. Given $\{\sigma_1, \cdots, \sigma_n\}$ since $f\in A^\infty$ it is not hard to prove that there is a τ, $\tau\geqq\sigma_i$ each i and a $L_j^\tau \in A_\tau$ such that $\varphi_{\sigma_i\tau}(L_j^\tau)=f(\sigma_i)$. But then $L_j^\tau\subseteq f(\sigma_i), i\leqq n$. Since L_j^τ is a local system, $\bigcap \{f(\sigma_i)|i\leqq n\}$ is a local system.

By a slight modification of a proof in [6, p. 132] we can show that Theorem M$\Rightarrow$Theorem H.

INDICATION OF PROOF.   We are assuming the hypotheses of Theorem H. For $x \in S$ let $\{f_H(x)|H \in L\}=\{u_1, \cdots, u_n\}$. We define the following partition σ_x of L.

$$L_{n+1}^{\sigma_x} = \{H \in L \mid x \notin H\},$$
$$L_i^{\sigma_x} = \{H \in L \mid x \in H\ \&\ f_H(x) = u_i\}, \qquad 1 \leqq i \leqq n.$$

Note that $L_{n+1}^{\sigma_x}$ is not a local system. By Theorem M we have a Malcev vector g for L. Define $f:S\to F$ by $f(x)=u_i$ iff $g(\sigma_x)=L_i^{\sigma_x}$. One can check that f has the desired property.

3. **Final comments.**   In [6] Theorem M is proven by a Zorn's lemma argument and in [2] Theorem K's proof uses the well-ordering theorem. We have shown that in Zermelo-Fraenkel set theory with only the axiom of choice for families of finite sets Theorems H, K, and M are equivalent. Thus these three technical results which have proven useful in obtaining local theorems are for all intents and purposes one and the same theorem. For those interested in the fine points of axiomatics, one can also show in the above mentioned version of set theory that the Boolean prime ideal theorem is equivalent to Theorems H, K, and M. This is of interest since the Boolean prime ideal theorem is strictly weaker than the axiom of choice (see Halpern and Lévy [1]).

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