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THE CONVERGENCE ALMOST EVERYWHERE OF LEGENDRE SERIES

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ABSTRACT. It is proved that the Legendre series of an L^p function converges almost everywhere, provided $4/3<p<\infty$. The result fails if $1\leqq p<4/3$.

It is a classical theorem of Marcel Riesz [1] that the Fourier series of an L^p function converges to it in the pth mean, provided that p exceeds 1. A similar result is true for Legendre series, provided that $4/3<p<4$ [2], but not otherwise [3].

Recently, R. A. Hunt [4], extending the work of Carleson, has shown that if $f\in L^p$, $p>1$, then its Fourier series converges p.p. By combining his theorem with standard equiconvergence theorems we can prove the first part of the following result.

THEOREM. *If $f\in L^p$ for some p in the range $4/3<p<\infty$, then its Legendre series converges p.p. The result fails if* $1\leqq p<4/3$.

It is interesting to contrast the range $4/3<p<\infty$ with the range $4/3<p<4$ of mean convergence.

The second part of the theorem follows from the fact that the Legendre series of $(1-x)^{-3/4}$ diverges everywhere [5, p. 249]. Incidentally, I do not know what happens if $p=4/3$; analogy with the Fourier case suggests failure of the result there.

We turn to the first part of the theorem, and *assume that* $4/3<p\leqq 2$. This is clearly no handicap, for if $f\in L^p$ for some p greater than 2 it also belongs to L^2. Because $f\in L^p$ for some p greater than $4/3$, it follows from Hölder's inequality that

$$\int_{-1}^{1}(1-x^2)^{-1/4}\,|f(x)|\,dx<\infty. \tag{1}$$

This enables us to invoke an equiconvergence theorem of Szegö [5, p. 239] which says this: let $s_n(x)$ denote the partial sums of the Legendre series of

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$f(x)$, and let $S_n(\theta)$ denote the partial sums of the cosine series of

$$g(\theta) = (\sin\theta)^{1/2} f(\cos\theta), \qquad 0 \leqq \theta \leqq \pi. \tag{2}$$

Then, under condition (1),

$$\lim_{n\to\infty} [s_n(\cos\theta) - (\sin\theta)^{-1/2} S_n(\theta)] = 0, \qquad 0 < \theta < \pi. \tag{3}$$

We shall show shortly that $g(\theta) \in L^q(0, \pi)$ for some q greater than 1. Then, according to Hunt's theorem [**4**],

$$\lim_{n\to\infty} S_n(\theta) = g(\theta) \quad \text{p.p.}$$

From this and (3) we conclude that $\lim_{n\to\infty} S_n(\cos\theta) = f(\cos\theta)$ p.p. This establishes the theorem.

It remains to show that $g(\theta)$, defined by (2), belongs to L^q for some q greater than 1. Writing $u = \cos\theta$, this means that we are to show that

$$\int_{-1}^{1} (1 - u^2)^{\beta} |f(u)|^q \, du < \infty \tag{4}$$

where $\beta = q/4 - 1/2$. We shall choose

$$q = (1/2)(4 + p)/(4 - p). \tag{5}$$

Because $4/3 < p \leqq 2$ it is easy to verify that $1 < q < p$. Now let $\alpha = p/q$, $\alpha' = p/(p-q)$. According to Hölder's inequality the integral in (4) is bounded by

$$\left(\int_{-1}^{1} (1 - u^2)^{\beta\alpha'} \, du\right)^{1/\alpha'} \left(\int_{-1}^{1} |f(u)|^p \, du\right)^{1/\alpha}$$

We are done if $\beta\alpha' > -1$, i.e. if

$$(1/2 - q/4)(p/(p - q)) < 1. \tag{6}$$

To prove (6) start with $p > 4/3$. According to (5) this fact can be written $q(4-p) < 2p$. Divide by $4p$ to obtain successively

$$q(1/p - 1/4) < 1/2, \qquad 1/2 - q/4 < 1 - q/p = (p - q)/p,$$

from which (6) follows.

Similar results can be obtained for Jacobi series using the general form of Szegö's equiconvergence theorem [**5**, p. 239] and his counterexamples [**5**, p. 249]. Corresponding results for Laguerre and Hermite series are given in [**6**].

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