THE CONVERGENCE ALMOST EVERYWHERE
OF LEGENDRE SERIES

HARRY POLLARD

Abstract. It is proved that the Legendre series of an \( f \in L^p \) function converges almost everywhere, provided \( 4/3 < p < \infty \). The result fails if \( 1 \leq p < 4/3 \).

It is a classical theorem of Marcel Riesz [1] that the Fourier series of an \( L^p \) function converges to it in the \( p \)th mean, provided that \( p \) exceeds 1. A similar result is true for Legendre series, provided that \( 4/3 < p < 4 \) [2], but not otherwise [3].

Recently, R A. Hunt [4], extending the work of Carleson, has shown that if \( f \in L^p, p > 1 \), then its Fourier series converges p.p. By combining his theorem with standard equiconvergence theorems we can prove the first part of the following result.

Theorem. If \( f \in L^p \) for some \( p \) in the range \( 4/3 < p < \infty \), then its Legendre series converges p.p. The result fails if \( 1 \leq p < 4/3 \).

It is interesting to contrast the range \( 4/3 < p < \infty \) with the range \( 4/3 < p < 4 \) of mean convergence.

The second part of the theorem follows from the fact that the Legendre series of \( (1 - x)^{-1/4} \) diverges everywhere [5, p. 249]. Incidentally, I do not know what happens if \( p = 4/3 \); analogy with the Fourier case suggests failure of the result there.

We turn to the first part of the theorem, and assume that \( 4/3 < p < 2 \). This is clearly no handicap, for if \( f \in L^p \) for some \( p \) greater than 2 it also belongs to \( L^2 \). Because \( f \in L^p \) for some \( p \) greater than \( 4/3 \), it follows from Hölder’s inequality that

\[
\int_{-1}^{1} (1 - x^2)^{-1/4} |f(x)| \, dx < \infty.
\]

This enables us to invoke an equiconvergence theorem of Szegö [5, p. 239] which says this: let \( s_n(x) \) denote the partial sums of the Legendre series of
Let $f(x)$, and let $S_n(\theta)$ denote the partial sums of the cosine series of

$$g(\theta) = (\sin \theta)^{1/2} f(\cos \theta), \quad 0 \leq \theta \leq \pi.$$ \hfill (2)

Then, under condition (1),

$$\lim_{n \to \infty} [s_n(\cos \theta) - (\sin \theta)^{-1/2} S_n(\theta)] = 0, \quad 0 < \theta < \pi.$$ \hfill (3)

We shall show shortly that $g(\theta) \in L^q(0, \pi)$ for some $q$ greater than 1. Then, according to Hunt’s theorem [4],

$$\lim_{n \to \infty} S_n(\theta) = g(\theta) \quad \text{p.p.}$$

From this and (3) we conclude that $\lim_{n \to \infty} S_n(\cos \theta) = f(\cos \theta) \text{ p.p.}$ This establishes the theorem.

It remains to show that $g(\theta)$, defined by (2), belongs to $L^q$ for some $q$ greater than 1. Writing $u = \cos \theta$, this means that we are to show that

$$\int_1^1 (1 - u^2)^{\beta} |f(u)|^q du < \infty$$ \hfill (4)

where $\beta = q/(4 - 1/2)$. We shall choose

$$q = (1/2)(4 + p)/(4 - p).$$ \hfill (5)

Because $4/3 < p \leq 2$ it is easy to verify that $1 < q < p$. Now let $\alpha = p/q$, $\alpha' = p/(p - q)$. According to Hölder’s inequality the integral in (4) is bounded by

$$\left( \int_1^1 (1 - u^2)^{\beta \alpha'} du \right)^\frac{1}{\beta} \left( \int_1^1 |f(u)|^q du \right)^\frac{1}{\alpha}$$

We are done if $\beta \alpha' < -1$, i.e. if

$$q(1/2 - q/4)(p/(p - q)) < 1.$$ \hfill (6)

To prove (6) start with $p > 4/3$. According to (5) this fact can be written $q(4 - p) < 2p$. Divide by $4p$ to obtain successively

$$q(1/p - 1/4) < 1/2, \quad 1/2 - q/4 < 1 - q/p = (p - q)/p,$$

from which (6) follows.

Similar results can be obtained for Jacobi series using the general form of Szegö’s equiconvergence theorem [5, p. 239] and his counterexamples [5, p. 249]. Corresponding results for Laguerre and Hermite series are given in [6]. *
REFERENCES


DIVISION OF MATHEMATICAL SCIENCES, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907