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SOME PROPERTIES OF SPECIAL FUNCTIONS DERIVED FROM THE THEORY OF CONTINUOUS TRANSFORMATION GROUPS

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ABSTRACT. The theory of continuous transformation groups is utilized in the study of some properties of special functions. On constructing the continuous transformation groups corresponding to a suitably defined infinitesimal transformation, a bilateral generating relation involving Laguerre polynomials $\{L_n^{(\alpha)}(x)\}$ is obtained in §2. It is shown to be a generalisation of Brafman's result. In the last section raising and lowering operators for $\{P_n^{(\alpha,\beta-n)}(x)\}$ and their commutator are introduced and on showing that they generate a 3-dimensional Lie algebra, the idea of c.t. groups is employed to establish a generating relation involving $\{P_n^{(\alpha,\beta-n)}(x)\}$ which is seen to yield a number of known results. Moreover, a bilateral generating relation involving $\{P_n^{(\alpha,\beta-n)}(x)\}$ is obtained; this is seen to be a generalisation of a well-known relation due to Weisner.

1. The purpose of this note is to prove some relations involving the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}$ and the Jacobi polynomials $\{P_n^{\alpha,\beta-n}(x)\}$ by the methods from the theory of continuous transformation groups. (For previous works using these methods see **[1]**, **[2]**, **[3]**, **[4]**, **[5]**.) Our principal results are contained in (2.6), (2.8), (3.7), (3.8) and (3.12).

2. The Laguerre polynomial $L_n^{(\alpha)}(x)$ is defined by

$$L_n^{(\alpha)}(x) = ((1 + \alpha)_n/n!)\,{}_1F_1(-n;\, 1 + \alpha;\, x).$$

Let us consider the function

$$(2.1)\qquad F_n(x, t, z) = \exp(n + \tfrac{1}{2}(\alpha + 1)t + z - \tfrac{1}{2}x)\cdot x^{(\alpha+1)/2}L_n^{(\alpha)}(x).$$

By the help of the well-known recursion relations for $\{L_n^{(\alpha)}(x)\}$, it can be easily checked that the operator $\mathscr{R}=e^t(\partial/\partial t-\tfrac{1}{2}x\partial/\partial z-x(\partial/\partial x))$ satisfies

$$(2.2)\qquad \mathscr{R}F_n(x, t, z) = (n + 1)F_{n+1}(x, t, z).$$

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The effects of $(\exp w\mathscr{R})$ upon x, t, z are seen to be [3]

$$(2.3)\qquad (\exp w\mathscr{R})\begin{Bmatrix} x \\ t \\ z \end{Bmatrix} = \begin{Bmatrix} x' \\ t' \\ z' \end{Bmatrix} = \begin{Bmatrix} x/(1-we^t) \\ t-\log(1-we^t) \\ z-xwe^t/2(1-we^t) \end{Bmatrix}.$$

Varma recently proved that [6]

$$(2.4)\qquad \begin{aligned}(1-w)^{-a}\phi_1\left(a;b,b+\alpha+1;\frac{w}{w-1},\frac{wx}{w-1}\right)\\ =\sum_{n=0}^{\infty}\frac{(a)_n}{(b+\alpha+1)_n}w^nL_n^{(\alpha)}(x),\end{aligned}$$

where $|w/(w-1)|<1$, $\alpha>-1$, and $\phi_1(\alpha;\beta,\gamma;x,y)$ is the confluent hypergeometric series

$$\phi_1(\alpha;\beta,\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_n}{(\gamma)_{m+n}m!\,n!}x^ny^m.$$

In (2.4), we replace w by wye^t and multiply both sides by

$$\exp\{\tfrac{1}{2}(\alpha+1)t+z-\tfrac{1}{2}x\}\cdot x^{(\alpha+1)/2},$$

so that by (2.1), we have

$$\begin{aligned}&(1-wye^t)^{-a}\exp\{\tfrac{1}{2}(\alpha+1)t+z-\tfrac{1}{2}x\}\cdot x^{(\alpha+1)/2}\\ &\qquad\times\phi_1(a;b,b+\alpha+1;wye^t/(wye^t-1),xwye^t/(wye^t-1))\\ &\qquad=\sum_{n=0}^{\infty}\frac{(a)_n}{(b+\alpha+1)_n}w^ny^nF_n(x,t,z).\end{aligned}$$

On operating both sides of the last equation by $\exp w\mathscr{R}$ and recalling (2.2) and (2.3), we have

$$\begin{aligned}&(1-wye^{t'})^{-a}\exp\{\tfrac{1}{2}(\alpha+1)t'+z'-\tfrac{1}{2}x'\}\cdot x'^{(\alpha+1)/2}\\ &\qquad\times\phi_1(a;b,b+\alpha+1;wye^{t'}/(wye^{t'}-1),x'wye^{t'}/(wye^{t'}-1))\\ &\qquad=(\exp w\mathscr{R})\sum_{n=0}^{\infty}\frac{(a)_n}{(b+\alpha+1)_n}w^ny^nF_n(x,t,z)\\ &\qquad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_n}{(b+\alpha+1)_n}y^n\frac{w^{m+n}}{m!}\mathscr{R}^mF_n(x,t,z)\\ &\qquad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{n!\,m!}\cdot\frac{(a)_n}{(b+\alpha+1)_n}y^nw^{m+n}F_{m+n}(x,t,z)\\ &\qquad=\sum_{n=0}^{\infty}w^nF_n(x,t,z)\sum_{m=0}^{n}\binom{n}{m}\frac{(a)_m}{(b+\alpha+1)_m}y^m\end{aligned}$$

$$(2.5)\qquad =\sum_{n=0}^{\infty}w^nf_n(y)F_n(x,t,z)$$

where

$$f_n(y) = \sum_{m=0}^{n} \binom{n}{m} \frac{(a)_m}{(b+\alpha+1)_m} y^m = {}_2F_1\left[\begin{matrix} -n, a; \\ b+\alpha+1; \end{matrix} -y\right].$$

On substituting the values of x', t', z' in (2.5) and on putting $t=z=0$, the relation (2.5) becomes

$$\begin{aligned} &\sum_{n=0}^{\infty} w^n {}_2F_1\left[\begin{matrix} -n, a; \\ b+\alpha+1; \end{matrix} y\right] L_n^{(\alpha)}(x) \\ &\quad = \exp\left(\frac{-xw}{1-w}\right) \cdot (1-w)^{a-\alpha-1}(1-w+yw)^{-a} \\ &\qquad \times \phi_1(a; b, b+\alpha+1; \\ &\qquad\qquad yw/(1-w+yw), xyw/(1-w+wy)(1-w)), \end{aligned} \tag{2.6}$$

which is noteworthy.

This result may be considered as a generalisation of the following relation of Brafman [9]:

$$\begin{aligned} &\sum_{0}^{\infty} {}_2F_1\left[\begin{matrix} -n, c; \\ 1+\alpha; \end{matrix} y\right] L_n^{(\alpha)}(x) w^n \\ &\quad = \exp(-xw/(1-w)) \cdot (1-w)^{c-\alpha-1} \cdot (1-w+wy)^{-c} \\ &\qquad \times {}_1F_1\left[\begin{matrix} c; \\ 1+\alpha; \end{matrix} \frac{xyw}{(1-w+wy)(1-w)}\right]. \end{aligned}$$

We can easily obtain this from (2.6) by the substitution $b=0$.

Now we consider some more particular cases of (2.6). It is known that Jacobi polynomial $P_n^{(\mu,\nu)}(z)$ admits of the following explicit evaluation [7]

$$P_n^{(\mu,\nu)}(z) = \frac{(1+\mu)_n}{n!} {}_2F_1\left[\begin{matrix} -n, 1+\mu+\nu+n; \\ 1+\mu; \end{matrix} \frac{1-z}{2}\right]. \tag{2.7}$$

Let us put in (2.6) $b=\mu-\alpha$, $a=1+\mu+\nu$, $y=(1-z)/2$; then from (2.6) follows the bilateral generating relation stated below:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{n!\, w^n}{(1+\mu)_n} P_n^{(\mu,\nu-n)}(z) L_n^{(\alpha)}(x) \\ &\quad = \exp(-xw/(1-w))(1-w)^{\mu+\nu-\alpha}(1-\tfrac{1}{2}(1+z)w)^{-1-\mu-\nu} \\ &\qquad \times \phi_1\Big(1+\mu+\nu; \mu-\alpha, \mu+1; \\ &\qquad\qquad \frac{w(1-z)}{2-w-zw}, \frac{(1-z)xw}{(2-w-zw)(1-w)}\Big). \end{aligned} \tag{2.8}$$

This seems to be new.

We note that $\phi_1(\alpha;\beta,\gamma;x,y)={}_2F_1[{}^{\alpha,}{}_{\gamma}{}^{\beta};y]$ when $x=0$, and that $L_n^{(\alpha)}(0)=(1+\alpha)_n/n!$, and so the relation (2.8) becomes

$$\sum_{n=0}^{\infty}\frac{(1+\alpha)_n}{(1+\mu)_n}w^nP_n^{(\mu,\nu-n)}(z)=(1-w)^{\mu+\nu-\alpha}(1-\tfrac{1}{2}(1+z)w)^{-1-\mu-\nu}\times{}_2F_1\left[\begin{matrix}1+\mu+\nu,\mu-\alpha; \\ 1+\mu;\end{matrix}\ \frac{w(1-z)}{2-w-zw}\right]$$

which may be compared with the result due to Varma [**6**] stated in §3.

3. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n is defined in (2.7). In this section we shall find some generating relations for $P_n^{(\alpha,\beta-n)}(x)$.

For our convenience, we consider the sequence of polynomials $\{\phi_n(z)\}$ defined by

$$\phi_n(z)=\frac{(1+b)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,1+a; \\ 1+b;\end{matrix}\ -z\right]. \tag{3.1}$$

This sequence obeys the relations

$$[(a+1)z+(n+b+1)+z(z+1)(d/dz)]\phi_n(z)=(n+1)\phi_{n+1}(z),$$
$$[-n+z(d/dz)]\phi_n(z)=-(b+n)\phi_{n-1}(z).$$

The last two relations show that if we introduce the function $F_n(x,t,z)$: $F_n\equiv F_n(x,t,z)=\exp(y+nt)\phi_n(z)$ and the operators $\mathscr{R}$, $\mathscr{L}$:

(3.2a) $\mathscr{R}\equiv e^t[\{(a+1)z+(b+1)\}\partial/\partial y+\partial/\partial t+z(z+1)\partial/\partial z]$,

(3.2b) $\mathscr{L}\equiv e^{-t}[-\partial/\partial t+z\partial/\partial z]$,

then we obtain the following relations:

$$\mathscr{R}F_n=(n+1)F_{n+1},\qquad \mathscr{L}F_n=-(b+n)F_{n-1},\qquad \mathscr{L}F_0=0; \tag{3.3}$$

thus $\mathscr{R}$, $\mathscr{L}$ behaving as the raising and lowering operators for the system $\{F_n(x,t,z)\}$ respectively. We see that

$$[\mathscr{R}\mathscr{L}]=\mathscr{R}\mathscr{L}-\mathscr{L}\mathscr{R}=2\partial/\partial t+(b+1)\partial/\partial y\equiv X,$$
$$[X\mathscr{R}]=-2\mathscr{R},\qquad [X\mathscr{L}]=2\mathscr{L}.$$

Hence $\mathscr{R}$, $\mathscr{L}$, X are the basis of a 3-dimensional Lie algebra which is isomorphic to the Lie algebra of the 3-dimensional rotation group. Consequently, if $Y=p\mathscr{L}+q\mathscr{R}+rX$, then

$$\begin{aligned}\exp Y&=\exp(p\mathscr{L}+q\mathscr{R}+rX)\\&=(\exp u\mathscr{L})(\exp v\mathscr{R})(\exp \tau X)\end{aligned} \tag{3.4}$$

where u, v, τ are expressible as functions of p, q, r as follows [3]:

$$(3.5)\qquad \begin{aligned} e^{-\tau} &= \cosh s - (r/s)\sinh s,\\ u &= (p \tanh s)/(s - r \tanh s),\\ w &= (q/s)\cosh s \sinh s - (qr/s^2)\sinh^2 s, \qquad s = (r^2 - pq)^{1/2}. \end{aligned}$$

Now the continuous transformation groups corresponding to the infinitesimal transformations $\mathscr{R}$ and Y are

$$(\exp w\mathscr{R})\begin{Bmatrix} z\\ t\\ y\end{Bmatrix} = \begin{Bmatrix} z'\\ t'\\ y'\end{Bmatrix}$$

$$(3.6a)\qquad = \begin{Bmatrix} z/\{1-(z+1)we^t\}\\ t - \log(1 - we^t)\\ y-(b-a)\log(1-we^t)-(a+1)\log(1-we^t(1+z))\end{Bmatrix},$$

$$(\exp Y)\begin{Bmatrix} z\\ t\\ y\end{Bmatrix} = \begin{Bmatrix} z''\\ t''\\ y''\end{Bmatrix},$$

where

$$(3.6b)\qquad \begin{aligned} z'' &= z/(1 - ue^{-t})[1 - (z + 1 - ue^{-t})ve^t],\\ t'' &= t + \log(1 - ue^{-t}) - \log(1 + uv - ve^t) + 2\tau,\\ y'' &= y - (b - a)\log(1 + uv - ve^t)\\ &\quad - (a + 1)\log[1 - ve^t(1 - ue^{-t} + z)] + (b + 1)\tau, \end{aligned}$$

u, v, τ being the same as in (3.5).

Remembering the relation (3.4) and the fact that Y is an infinitesimal transformation, we apply the operator $\exp Y$ to the function $F_m(z, y, t)$:

$$\begin{aligned} F_m(z'', y'', t'') &= (\exp Y)F_m(z, y, t)\\ &= (\exp u\mathscr{L})(\exp v\mathscr{R})(\exp \tau X)F_m(z, y, t)\\ &= \exp \tau(2m + b + 1)\cdot(\exp u\mathscr{L})(\exp v\mathscr{R})\cdot F_m(z, y, t)\\ &= \exp \tau(2m + b + 1)\sum_{n=0}^{\infty}\binom{m+n}{n}v^n\sum_{k=0}^{m+n}\frac{u^k\mathscr{L}^k}{k!}F_{m+n}(z, y, t)\\ &= \exp \tau(2m + b + 1)\sum_{n=0}^{\infty}\binom{m+n}{n}v^n\\ &\quad\times\sum_{k=0}^{m+n}\binom{b+m+n}{k}(-u)^k\phi_{m+n-k}(z)\exp\{y + (m + n - k)t\} \end{aligned}$$

where $\binom{p}{k}$ stands for $p(p-1)\cdots(p-k+1)/k!$.

Putting $y=t=0$ and using (3.6b) we get

$$(1-u)^m(1+uv-v)^{-m-b+a}\{1-v(1-u+z)\}^{-a-1}$$
$$\times\ \phi_m(z/(1-u)(1-(1+z+u)v))$$
$$=\sum_{n=0}^{\infty}\binom{m+n}{n}v^n\sum_{k=0}^{m+n}(-u)^k\binom{b+m+n}{k}\phi_{m+n-k}(z).$$

We recall the definition of $\phi_n(z)$ in (3.1) and replace u by $-u$, so that the above relation gives

$$(1+u)^m(1-uv-v)^{-m-b+a}\{1-v(1+u-z)\}^{-a-1}$$
$$(3.7)\qquad \times\ {}_2F_1\left[\begin{matrix}-m,1+a;\\ 1+b;\end{matrix}\ \frac{z}{(1+u)(1-(1-z+u)v)}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(1+b)_{m+n}}{(1+b)_m}\cdot\frac{v^n}{n!}\sum_{k=0}^{m+n}u^k\binom{m+n}{k}{}_2F_1\left[\begin{matrix}-m-n+k,1+a;\\ 1+b;\end{matrix}\ z\right].$$

This is noteworthy.

Let us put $a=\alpha+\beta$, $b=\alpha$, $z=(1-x)/2$ in (3.7) and remember (3.1); then we have

$$(1+u)^m(1-uv-v)^{-m+\beta}\{1-v(u+(1+x)/2)\}^{-\alpha-\beta-1}$$
$$(3.8)\qquad \times\ P_m^{(\alpha,\beta-m)}(1-(1-x)/(1+u)(1-(u+(1+x)/2)v))$$
$$=\sum_{n=0}^{\infty}\binom{m+n}{n}v^n\sum_{k=0}^{m+n}\binom{\alpha+m+n}{k}u^kP_{m+n-k}^{(\alpha,\beta-m-n+k)}(x).$$

The formulae (3.7) and (3.8) are new. They contain previously derived results as special cases: the cases $u=0$ of (3.7), (3.8) are known, so are the cases $v=0$.

Next we note Varma's result [**6**]:

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(b+\alpha+1)_n}w^nP_n^{(\alpha,\beta-n)}(x)$$
$$=(1-w)^{-a}F_1\left(a,b,1+\alpha+\beta,1+\alpha+b;\frac{-w}{1-w},\frac{-w(1-x)}{2(1-w)}\right),$$

which we can write as follows:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(b+\beta+1)_n}w^n\phi_n(z)$$
$$=(1-w)^{-\alpha}F_1\left(\alpha,\beta,1+a,1+b+\beta;\frac{-w}{1-w},\frac{wz}{1-w}\right).$$

Replacing w by xwe^t and multiplying both sides by e^y, we have

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(b+\beta+1)_n} w^n x^n F_n(z, y, t)$$
$$(3.9) \quad = e^y(1 - xwe^t)^{-\alpha} F_1\left(\alpha, \beta, 1+a, 1+b+\beta; \frac{-xwe^t}{1-xwe^t}, \frac{xwze^t}{1-xwe^t}\right).$$

Operating both sides of (3.9) by $(\exp w\mathscr{R})$, we get, by virtue of (3.3),

$$(\exp w\mathscr{R}) \cdot e^y(1 - xwe^t)^{-\alpha}$$
$$\times F_1\left(\alpha, \beta, 1+a, 1+b+\beta; \frac{-xwe^t}{1-xwe^t}, \frac{xwe^t z}{1-xwe^t}\right)$$
$$= (\exp w\mathscr{R}) \cdot \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(b+\beta+1)_n} w^n x^n F_n(z, y, t)$$
$$(3.10) \quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_n w^{n+m} x^n}{(b+\beta+1)_n} \cdot \frac{\mathscr{R}^m}{m!} F_n(z, y, t)$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{w^m x^n (\alpha)_n}{(b+\beta+1)_n} \cdot \frac{m!}{n!\,(n-m)!} F_m(z, y, t)$$
$$= \sum_{m=0}^{\infty} w^m F_m(z, y, t) f_m(x)$$

where

$$f_m(x) = \sum_{n=0}^{\infty} \binom{m}{n} \frac{(\alpha)_n}{(b+\beta+1)_n} x^n$$
$$(3.11) \quad = \sum_{n=0}^{\infty} \frac{(-m)_n(\alpha)_n}{(b+\beta+1)_n} \cdot \frac{(-x)^n}{n!}$$
$$= {}_2F_1\left[\begin{matrix} -m, \alpha; \\ b+\beta+1; \end{matrix} -x\right].$$

Thus from (3.10), (3.11), (3.2) and (3.6a), we have

$$\sum_{m=0}^{\infty} \frac{(1+b)_m}{m!} \exp(y+mt) w^m {}_2F_1\left[\begin{matrix} -m, 1+a; \\ 1+b; \end{matrix} -z\right] {}_2F_1\left[\begin{matrix} -m, \alpha; \\ b+\beta+1; \end{matrix} -x\right]$$
$$= e^{y'}(1 - xwe^{t'})^{-\alpha} F_1\left(\alpha, \beta, 1+a, 1+b+\beta; \frac{-xwe^{t'}}{1-xwe^{t'}}, \frac{xwz'e^{t'}}{1-xwe^{t'}}\right)$$

which becomes, on substituting the values of z', y', t' from (3.6a) and thereafter putting $y=t=0$ and replacing x, z by $-x$, $-z$ respectively

$$\sum_{m=0}^{\infty} \frac{(1+b)_m}{m!} w^m {}_2F_1\left[\begin{matrix}-m, 1+a; \\ 1+b;\end{matrix} z\right] {}_2F_1\left[\begin{matrix}-m, \alpha; \\ b+\beta+1;\end{matrix} x\right]$$

$$= (1-w)^{a-b+\alpha}(1-w+wz)^{-a-1}(1-w+wx)^{-\alpha}$$

(3.12)
$$\times F_1\left(\alpha, \beta, 1+a, 1+b+\beta; \frac{xw}{1-w+wx}, \frac{xwz}{(1-w+wz)(1-w+wx)}\right).$$

A particular case of the relation (3.12) is noteworthy: putting $\beta=0$ and changing a, b to $a-1$, $b-1$ respectively, we get the following result proved by Weisner [**8**]:

$$\sum_{m=0}^{\infty} \frac{(b)_m}{m!} w^m {}_2F_1\left[\begin{matrix}-m, a; \\ b;\end{matrix} z\right] {}_2F_1\left[\begin{matrix}-m, \alpha; \\ b;\end{matrix} x\right]$$

$$= (1-w)^{a-b+\alpha}(1-w+wz)^{-a}(1-w+wx)^{-\alpha} \times {}_2F_1\left[\begin{matrix}a, \alpha; \\ b;\end{matrix} \frac{xzw}{(1-w+wz)(1-w+wx)}\right].$$

Thus our result (3.12) may be considered to be a generalisation of this result of Weisner.

Making a simple change of parameters and also of the variables x, z we see that the following relation is evident from (3.12):

$$\sum_{m=0}^{\infty} \frac{m!}{(1+\beta)_m} w^m P_m^{(b,a-m)}(z) P_m^{(\beta,\alpha-m)}(x)$$

$$= (1-w)^{a+\alpha+\beta+1}\{1-(1+z)w/2\}^{-a-b-1}\{1-(1+x)w/2\}^{-\alpha-\beta-1}$$

$$\times F_1\left(1+\alpha+\beta, \beta-b, 1+a+b, 1+\beta; \frac{(1-x)w/2}{1-(1+x)w/2}, \frac{(1-x)(1-z)w/4}{(1-(1+x)w/2)(1-(1+z)w/2)}\right).$$

This may be compared with a result due to Srivastava [**10**].

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