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GAUGE FUNCTIONS AND LIMIT SETS FOR NONAUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

H. K. WILSON

ABSTRACT. A gauge function V for the differential equation (S) $x'=f(x,t)$ is a scalar-valued function sufficiently smooth for $dV(\phi(t),t)/dt$ to exist almost everywhere for solutions $x=\phi(t)$, $t_0 \leqq t < \tau_\phi^+$, of (S). Let (S) have gauge function V that satisfies the following conditions: (1) $\lim_{t\to+\infty} V(x,t) \equiv \lambda(x)$ exists; (2) V is continuous in x uniformly with respect to t; (3) the upper, right derivate of V with respect to (S) is nonpositive. Then, if a solution $x=\phi(t)$ of (S) has an ω-limit point p, there is a unique constant $c(\phi)$ such that $\lambda(p)=c(\phi)$. An application to second order, linear equations is given.

1. **Context of the discussion.** Let f be continuous on a cylinder $D=P\times I$, where P is a domain in R^n and I is an interval $[\beta, +\infty)$, and consider the differential equation

$$\text{(S)} \qquad x' = f(x,t).$$

The fundamental qualitative problem for (S) is the description of the asymptotic behavior of its solutions. In this note, there is developed a technique for such description by means of gauge functions.

Let $Q\subset P$ be open. A *gauge function* for (S) is a function $V: Q\times I \to R^1$ that is bounded below on every cylinder $K\times I$ with compact base $K\subset Q$ and satisfies the following condition: For every compact neighborhood $K\times[a,b]$ in $Q\times I$, there is a number $L>0$ and an absolutely continuous function ψ such that

$$|V(x,t)-V(y,t)| \leqq L\cdot|x-y| \quad \text{and} \quad |V(x,s)-V(x,t)| \leqq |\psi(s)-\psi(t)|$$

for all $x, y\in K$ and $s, t\in[a,b]$. The *derivative DV of a gauge function V with respect to the equation* (S) is defined by the relation

$$DV(x,t) = \limsup_{h\to 0+} \frac{V(x+hf(x,t), t+h) - V(x,t)}{h}.$$

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Throughout the rest of the paper, consider as given a solution $x=\phi(t)$ of (S) that initiates at a point (x_0, t_0) in D, has escape time τ_ϕ^+ from P, and has at least one ω-limit point in P. Further, suppose that $\phi(t)$ is in a subset G of P for $t_0 \leqq t < \tau_\phi^+$ and that $V: Q \times I \to R^1$ is a gauge function for (S) with $Q \supset \bar{G} \cap P$.

Below, we establish circumstances under which the ω-limit set Ω_ϕ of ϕ is a subset of the locus of precisely one equation

$$\lim_{t \to +\infty} V(x, t) \equiv \text{constant}.$$

Then, we illustrate the applicability of the results by solving a stability problem for a linear oscillator with one degree of freedom.

2. **The results.** It is a routine matter to establish that $V(\phi(t), t)$ is locally absolutely continuous. Consequently, $dV(\phi(t), t)/dt$ exists for almost all $t \in [t_0, \tau_\phi^+)$ and equals $DV(\phi(t), t)$ when it does exist. Further, $V(\phi(t), t)$ is nonincreasing if $DV(\phi(t), t) \leqq 0$ for almost all t.

THEOREM 1. *Suppose that A is a nonempty subset of Q such that $DV(x, t) \leqq 0$ for (x, t) in the cylinder $A \times I$ and let the measure of $\{t: t \in [t_0, +\infty)$ and $\phi(t) \notin A\}$ be zero. Then there is a number v_ϕ with the following property: If $\{t_k\}$ is any sequence such that $t_k \to \tau_\phi^+$ and $\{\phi(t_k)\}$ converges to a point in P as $k \to +\infty$, then $V(\phi(t_k), t_k) \to v_\phi$ as $k \to +\infty$.*

PROOF. It is well known that $\Omega_\phi \cap P \neq \varnothing$ implies that $\tau_\phi^+ = +\infty$; see [1]. Let $\{t_k\}$ be such that $\phi(t_k) \to p \in P$ and $t_k \to +\infty$ as $k \to +\infty$ and let $\{s_m\}$ be such that $\phi(s_m) \to q \in P$ and $s_m \to +\infty$ as $m \to +\infty$. Then $p, q \in Q$ and the sequences $\{V(\phi(t_k), t_k)\}$ and $\{V(\phi(s_m), s_m)\}$ are nonincreasing for all sufficiently large values of the indices. They are bounded below since V is a gauge function. Denote the respective limits by τ and σ and extract subsequences $\{t_{k_i}\}$ and $\{s_{m_i}\}$ from $\{t_k\}$ and $\{s_m\}$ in such a way that $t_{k_i} \leqq s_{m_i} \leqq t_{k_{i+1}}$. Then

$$V(\phi(t_{k_{i+1}}), t_{k_{i+1}}) \leqq V(\phi(s_{m_i}), s_{m_i}) \leqq V(\phi(t_{k_i}), t_{k_i})$$

for all sufficiently large i and, passing to the limit, we have $\tau \leqq \sigma \leqq \tau$.

THEOREM 2. *Let the hypotheses of Theorem 1 hold. In addition, suppose that $\lim_{t \to +\infty} V(x, t) = \lambda(x)$ exists for every $x \in Q$ and that V is continuous at each $x \in Q$, uniformly with respect to t in I. Then, if $p \in \Omega_\phi$, $V(p, t) \to v_\phi$ as $t \to +\infty$.*

PROOF. Choose any sequence $\{t_k\}$ such that $\phi(t_k) \to p$ and $t_k \to +\infty$ as $k \to +\infty$ and let $\varepsilon > 0$ be given. There are numbers T and δ such that

$|V(p, t)-\lambda(p)|<\varepsilon/2$ for all $t\geqq T$ and, if $|x-p|<\delta$, then $|V(x, t)-V(p, t)|<\varepsilon/2$ for all t in I. Choose K so large that $k\geqq K$ implies $t_k\geqq T$ and $|\phi(t_k)-p|<\delta$. Then $k\geqq K$ also implies that

$$|\lambda(p) - V(\phi(t_k), t_k)| \leqq |\lambda(p) - V(p, t_k)| + |V(p, t_k) - V(\phi(t_k), t_k)| < \varepsilon.$$

Thus $\lambda(p)=v_\phi$.

COROLLARY. *The limit set Ω_ϕ is a subset of the locus of exactly one of the equations* $\lim_{t\to+\infty} V(x, t)\equiv$*constant.*

3. **An example.** Consider the linear second order equation

$$\text{(L)} \qquad x'' + q(t)x' + p(t)x = 0,$$

where q is nonnegative and continuous and p is absolutely continuous, nonincreasing and bounded below with $\lim_{t\to+\infty} p(t)=a$.

Any solution $x=\phi(t)$, $y=\phi'(t)$ of (L) will have for its ω-limit set a subset, possibly empty, of the locus of an equation $y^2+ax^2\equiv$*constant.* This is immediately established with the aid of the gauge function $V(x, y, t)=y^2+p(t)x^2$ and the corollary to Theorem 2 since $DV(x, y, t)=-2q(t)y^2+p'(t)x^2$.

If $a\leqq 0$, every ω-limit set is a subset of a hyperbola or a pair of lines in the xy-plane.

Suppose that $a>0$ and let

$$\text{(T)} \qquad x = \phi(t), \qquad y = \phi'(t)$$

denote an arbitrary solution of (L). We assert that (T) approaches precisely one (possibly degenerate) ellipse $y^2+ax^2=c\geqq 0$.

To prove the assertion, assume first, for contradiction, that the ω-limit set Ω_ϕ of (T) is empty. The $|\phi(t)|+|\phi'(t)|\to+\infty$ as $t\to+\infty$ for, otherwise, $(\phi(t), \phi'(t))$ would be interior to some ball $x^2+y^2<r^2$ for an unbounded sequence of times and the Bolzano-Weierstrass theorem would imply that Ω_ϕ is not empty. Choose an $M>0$ such that $(\phi(t), \phi'(t))$ is exterior to the ellipse $x^2+ay^2=M$ for all sufficiently large t. Then

$$V(\phi(t), \phi'(t), t) \geqq [\phi'(t)]^2 + a\phi^2(t) \geqq M$$

for such t and, since $V(\phi(t), \phi'(t), t)$ is nonincreasing, it has a limit b as $t\to+\infty$. It follows that $[\phi'(t)]^2+a\phi^2(t)\leqq b+1$ for all sufficiently large t, consequently (T) is bounded as $t\to+\infty$, a contradiction. Thus Ω_ϕ is not empty.

By the corollary to Theorem 2, Ω_ϕ lies in precisely one ellipse $y^2+ax^2=c\geqq 0$. The solution (T) must approach the ellipse as $t\to+\infty$ because Ω_ϕ would be disconnected [2] if (T) did not.

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Department of Mathematical Studies, Southern Illinois University, Edwardsville, Illinois 62025