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A NOTE ON KAPLUN LIMITS AND DOUBLE ASYMPTOTICS

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ABSTRACT. The foundation of "inner-outer" expansions and allied methods is examined. Two limit definitions (both yielding Kaplun's extension theorem) are compared in regard to double limit asymptotics.

1. **Introduction.** The great practical usefulness ([**1**]–[**4**]) of "inner-outer" or "matched asymptotic" approximations and allied methods has created interest ([**5**]–[**7**]) in the mathematical concept of Kaplun limits. The following is an attempt to clarify the concept and some difficulties associated with it. Two definitions (§2) will be compared: that of [**6**] and a broader one which may, perhaps, be closer to that envisioned by Kaplun [**5**]. It is first established that Kaplun's extension theorem holds for the latter, so that the definition is consistent with the techniques of matched asymptotic expansions. The applicability of the limit process is not limited to matched asymptotic expansions, however; it has been applied to nonlinear problems of physical importance ([**6**], [**8**]) involving asymptotic approximations to solutions valid for large values of a variable and small values of a parameter. The limit definition presented here is formulated in order to establish a useful connection between such double limits and suitable families of Kaplun limits in which the variable and parameter tend to their respective limits simultaneously. The connection is found to necessitate the understanding of the sense of approximation in an appropriate manner.

2. **Kaplun limits.** Let ε be a positive real parameter, and x a variable in $\boldsymbol{R}^n$ with Euclidean norm $|x|$. Let F be a function, defined for $\varepsilon \in (0, 1]$ and on some domain of x-space with pointwise norm $\|F\|$. Typically, F might be a vector-valued function defined by a system of differential equations, and our interest is in the limit $\varepsilon\downarrow 0$. This limit may be singular; for example, it may result in the loss of the highest order derivatives in a

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differential equation, and a corresponding inability to satisfy all the boundary conditions. For simplicity we will assume that $x=0$ is the only singularity, and then the essential idea of the Kaplun limit is to study the limit $\varepsilon\downarrow 0$ not for fixed x near $x=0$, but for x tending to the singular point in a definite relationship to ε, specified by a "stretching" transformation

$$x_\delta = x/\delta(\varepsilon), \qquad G(x_\delta;\varepsilon) = F(x;\varepsilon),$$

with $\delta(\varepsilon)$ a positive continuous function on $(0, 1]$. (To avoid prohibitive generality, it is assumed that near $x=0$, F depends primarily on $|x|$, so that the single stretching scale $\delta(\varepsilon)$ suffices.) Two definitions of the limit process follow.

Definition A. If the function $G(x_\delta; +0)\equiv\lim_{\varepsilon\downarrow 0} G(x_\delta;\varepsilon)$ exists uniformly on $\{x_\delta \mid |x_\delta|>0\}$, then define $\lim_\delta F(x;\varepsilon)=G(x_\delta; +0)$. ("Uniformly" refers to approach to the limit uniformly for $|x_\delta|>0$ **[6]**.)

Definition B. If for all closed intervals $[y_1, y_2]\subset(0,\infty)$, the function $G(x_\delta; +0)\equiv\lim_{\varepsilon\downarrow 0} G(x_\delta;\varepsilon)$ exists and is approached uniformly on the set $\{x_\delta \mid |x_\delta|\in[y_1,y_2]\}$, then $\lim_\delta F(x;\varepsilon)\equiv G(x_\delta; +0)$.

Definition A is that of Meyer **[6]**. The less restrictive Definition B will be adopted in the following. There are indications in **[5]** that Kaplun had B rather than A in mind. The more informal notation

$$\lim_{\varepsilon\downarrow 0,\, x_\delta \text{ fixed}} F(x_\delta\delta(\varepsilon);\varepsilon),$$

which some readers may find easier, is also encountered.

Consider now the collection Ω of all positive continuous functions $\gamma(\varepsilon)$ of the single variable ε, defined on $(0, 1]$, and separate them into equivalence classes ord γ defined **[5]** by

$$\operatorname{ord}\gamma = \left\{\varphi(\varepsilon)\in\Omega \;\middle|\; \lim_{\varepsilon\downarrow 0}\frac{\varphi(\varepsilon)}{\gamma(\varepsilon)} \text{ exists and is different from } 0\right\}.$$

A partial ordering is constructed on these by defining $\operatorname{ord}\gamma_1<\operatorname{ord}\gamma_2$ if and only if $\lim_{\varepsilon\downarrow 0}(\gamma_1(\varepsilon)/\gamma_2(\varepsilon))=0$.

The division into equivalence classes is motivated by the expectation that the limit definition be essentially independent of the choice of representative of any class, so that if $\lim_\delta F$ exists and $\operatorname{ord}\gamma=\operatorname{ord}\varphi$, then $\lim_\gamma F=\lim_\delta F$. However, due to the explicit dependence of the function $\lim_\delta F$ on the stretched variable x_δ, the two limits can only be shown to have the same trace. For the application discussed here, the following weaker statement, proved in the appendix, is sufficient.

Lemma 1. *If* $\lim_\delta F=0$ *and* $\operatorname{ord}\gamma=\operatorname{ord}\delta$, *then* $\lim_\gamma F=0$.

3. **Extension theorem.** A topology can be introduced on the collection of order classes ([5], [6]); that of Meyer will be adopted here. A set S of order classes is said to be *convex* if ord δ_1, ord $\delta_2 \in S$, and ord $\delta_1 <$ ord $\theta <$ ord δ_2 together imply that ord $\theta \in S$. A set S is said to be *convex-open* if it is convex and if ord $\theta \in S$ implies the existence of elements ord γ, ord $\delta \in S$ such that ord $\gamma <$ ord $\theta <$ ord δ. It is readily shown that the convex-open sets constitute a basis for a topology on the order classes.

By contrast, a set S is said to be a *closed interval* if it is convex and possesses particular elements ord δ_1, ord δ_2 such that ord $\delta_1 \leqq$ ord θ and ord $\theta \leqq$ ord δ_2 for every ord $\theta \in S$. We will also have occasion to use the notation R_γ, the *right set* of ord γ [6], defined as $R_\gamma = \{\text{ord } \theta \mid \text{ord } \gamma < \text{ord } \theta\}$.

The structure of the space of order classes shows the number of different Kaplun limits to be very large. In many cases, however, they fall naturally into families such that the distinctions between different limits in the same family are not substantial [6]. This motivates the

Definition. $H(x;\varepsilon)$ is a $\Delta(\varepsilon)$-approximation to $F(x;\varepsilon)$ on a convex set S shall mean $\lim_\delta \|\Delta^{-1}(F-H)\| = 0$ for all ord $\delta \in S$.

That these definitions are consistent with the principle of matched asymptotic expansions is established by the following

Theorem (Extension). *If H is a Δ-approximation to $F(x;\varepsilon)$ on a closed interval S_0, then it is so also on an open set $S \supset S_0$.*

Proof. It suffices to prove the statement for sets S_0 consisting of a single order class, and by suitable renaming of the variable we may assume that $S_0 = \{\text{ord } 1\}$. Let I_n denote the interval $[1/n, n]$. Then the hypothesis is that there is a positive sequence ε_n such that $w \equiv \|\Delta^{-1}(F-H)\| < 1/n$ whenever $x \in I_n$ and $\varepsilon \leqq \varepsilon_n$. Without loss of generality, it may be assumed that $\{\varepsilon_n\}$ decreases monotonically to zero.

Let f and g be continuous monotone functions satisfying $f(\varepsilon_n) = 1/(n-1)$ and $g(\varepsilon_n) = n-1$. Clearly ord $f <$ ord $1 <$ ord g. Define

$$S = \{\text{ord } \theta \mid \text{ord } f < \text{ord } \theta < \text{ord } g\},$$

and let ord $\delta \in S$.

Now let the interval $I = [y_1, y_2] \subset (0, \infty)$ be given, with $x_\delta \in I$. Since ord $\delta \in S$, there exists an integer $N = N(I)$ such that for $\varepsilon < \varepsilon_N$, $\delta(\varepsilon) > (1/y_1)f(\varepsilon)$ and $\delta(\varepsilon) < (1/y_2)g(\varepsilon)$. For any $n > N$, let ε be given such that $0 < \varepsilon < \varepsilon_n$. Then there is an integer $p \geqq 0$ such that $\varepsilon_{n+p+1} \leqq \varepsilon \leqq \varepsilon_{n+p}$ since $\{\varepsilon_n\}$ decreases monotonically to zero. Hence

$$x = x_\delta \delta(\varepsilon) > y_1(1/y_1)f(\varepsilon) \geqq f(\varepsilon_{n+p+1}) = 1/(n+p)$$

and

$$x = x_\delta \delta(\varepsilon) < y_2(1/y_2)g(\varepsilon) \leqq g(\varepsilon_{n+p+1}) = n + p,$$

so that $x \in I_{n+p}$, while $\varepsilon \leqq \varepsilon_{n+p}$. Thus $w < 1/(n+p)$, and $\lim_\delta w = 0$.

4. Double limits. We now turn to a somewhat different context, involving the application of Kaplun limits to asymptotic approximation of solutions to nonlinear partial differential equations, valid for large values of one of the independent variables t and small values of a parameter ε. This has been discussed by Meyer in **[6]**. The situation envisaged is one in which the limit $\varepsilon \to 0$ yields a simpler problem for which the asymptotic behavior of solutions as $t \to \infty$ can readily be analysed, whereas the physically more meaningful sequence of the (generally noncommutative) limits is just the reverse, involving asymptotics in t for fixed, small positive values of ε. Indeed, in **[1]**, with t representing time and ε a small amplitude, or in **[8]**, with t representing distance and ε a wave number, the former sequence of limits has the effect of precluding all but trivial solutions, while the latter is not readily analysed. A possibly successful compromise is to evaluate the limits simultaneously, by the replacement of the $\lim_{\varepsilon \to 0} \lim_{t \to \infty}$ with a suitable family of Kaplun limits. Of course, the functions studied are not expected to tend to proper limits as $t \to \infty$, but rather to approach simpler functions of t, so that the following notation for asymptotic approximation will be useful.

DEFINITION. Write $\varphi = \lim^*_{x \to x_0} \psi$ *to order* Δ to indicate that $\varphi(x)$ is an asymptotic approximation to $\psi(x)$, correct to order $\Delta(x)$ inclusive, as $x \to x_0$; i.e., $\|\Delta^{-1}(\varphi - \psi)\| \to 0$ as $x \to x_0$. When no order is specified, it is to be understood that the approximation is correct to order unity.

The connection between the double limit and the Kaplun limit may then be phrased as:

LEMMA 2A. *If h is a $\Delta(\varepsilon)$-approximation to $f(t; \varepsilon)$ in a right set R_γ, then h is also an asymptotic approximation, correct to $O(\Delta)$ inclusive, to $\lim^*_{t \to \infty} f$ as $\varepsilon \to 0$, for all sufficiently large t.*

This is the form presented by Meyer, in conjunction with the more restrictive Definition A. With that definition, however, it can be shown (Appendix) that in fact h converges uniformly to f as $\varepsilon \to 0$ for all t greater than some constant $T_0 > 0$, and so the intermediary $\lim^* f$ is unnecessary. The alternative Definition B presented above was motivated by the desire to avoid this conclusion, which represents unusually good fortune in the selection of the approximating function h.

On the other hand, if Definition B is used, Lemma 2A is false. As a counterexample, consider $f(t; \varepsilon) = \exp(-\varepsilon t)$. An ε^n-approximation to f

is given by $h(t;\varepsilon)=\varepsilon^{n+1}$, valid in the right set of ord $1/\varepsilon$. For fixed $\varepsilon>0$, $\lim^*_{t\to\infty} f=t^{-(n+1)}$, correct to $O(t^{-n})$. But

$$\left\|\Delta^{-1}\left(h-\lim_{t\to\infty}{}^* f\right)\right\| = |\varepsilon^{-n}(\varepsilon^{n+1}-t^{-(n+1)})| = |\varepsilon-\varepsilon^{-n}t^{-(n+1)}|$$

and there is no $T_0>0$ such that the last expression tends to zero as $\varepsilon\to 0$ for fixed $t>T_0$.

To formulate a valid conjecture, we note that for the situation considered in [1], the limit $\varepsilon\downarrow 0$ for fixed t does not necessarily have direct physical relevance; no sequence of experiments with ε tending to zero will actually be carried out. The principal concern is to characterize the ultimate, as opposed to transient, behavior in time for fixed, small ε. Hence the significance of the right set for the approximation h is that two experimenters, using different values of the parameter ε, will both ultimately observe the behavior described by h; one may have to wait longer than the other.

This approximation concept may be expressed as:

DEFINITION. $h(t;\varepsilon)$ is an *eventual* $\Delta(\varepsilon)$*-approximation* to $\lim^*_{t\to\infty} f(t;\varepsilon)$ as $\varepsilon\to 0$ means that, given any $\alpha>0$, a number $A_\alpha>0$ and positive function $T_\alpha(\varepsilon)$ on $(0, A_\alpha]$ exist such that $0<\varepsilon<A_\alpha$ and $t>T_\alpha(\varepsilon)$ imply

$$\left\|\Delta^{-1}\left(h(t;\varepsilon)-\lim_{t\to\infty}{}^* f(t;\varepsilon)\right)\right\| < \alpha.$$

The relation between double limit and single Kaplun limit can then be formulated as:

LEMMA 2B. *$h(t;\varepsilon)$ is a $\Delta(\varepsilon)$-approximation to $f(t;\varepsilon)$ on a right set if, and only if, it is an eventual Δ-approximation to $\lim^*_{t\to\infty} f$ as $\varepsilon\to 0$.*

PROOF. See Appendix.

It is important to note that the approximation h is not an asymptotic approximation in the usual sense. For fixed ε, the approximation does not necessarily improve as $t\to\infty$.

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Appendix.

1. PROOF OF LEMMA 1. Let $[y_1, y_2]\subset(0,\infty)$. It must be shown that $F(x_\gamma\gamma(\varepsilon);\varepsilon)\to 0$ uniformly for $|x_\gamma|\in[y_1, y_2]$. Since ord $\gamma=$ ord δ, there exists a constant $B>0$ such that $x_\delta\equiv(\gamma(\varepsilon)/\delta(\varepsilon))x_\gamma\to Bx_\gamma$ as $\varepsilon\to 0$. Thus for some $\varepsilon_0>0$, $|x_\gamma|\in[y_1, y_2]\Rightarrow|x_\delta|\in[(B/2)y_1, (3B/2)y_2]$ for all $\varepsilon<\varepsilon_0$.

Since $\lim_\delta F=0$, $F(x_\delta\delta(\varepsilon);\varepsilon)=F(x_\gamma\gamma(\varepsilon);\varepsilon)\to 0$ uniformly for $|x_\delta|\in[(B/2)y_1,(3B/2)y_2]$, and hence for $|x_\gamma|\in[y_1,y_2]$.

2. LEMMA. *Under the conditions of Lemma* 2A (*and using Definition* A), *there exists* $T_0>0$ *such that* h *converges to* f *uniformly on* $t\geqq T_0$.

PROOF. The hypothesis is that $\lim_\theta \|\Delta^{-1}(f-h)\|=0$ for $\theta\in R_\gamma$.

Let $\delta\in R_\gamma$ be such that $\delta(\varepsilon)$ is positive, monotone increasing as $\varepsilon\downarrow 0$. Let $T_0=\delta(1)$, and let $w(t;\varepsilon)=\|\Delta^{-1}(f-h)\|$. Then, by Definition A, $\lim_\delta w=0$ implies the existence of a positive sequence A_n decreasing monotonically to zero as $n\to\infty$ such that given any $t_\delta>0$, $w(t_\delta\delta(\varepsilon);\varepsilon)<1/n$ if only $\varepsilon\leqq A_n$.

Now fix $t\geqq T_0$. Let $t_\delta(\varepsilon)=t/\delta(\varepsilon)$; $t_\delta(\varepsilon)>0$ since $\delta(\varepsilon)$ is positive. Hence $w(t;\varepsilon)=w(t_\delta(\varepsilon)\delta(\varepsilon);\varepsilon)<1/n$ if $\varepsilon\leqq A_n$, and it follows that $w(t;\varepsilon)\to 0$ as $\varepsilon\downarrow 0$, uniformly in $t\geqq T_0$.

3. PROOF OF LEMMA 2B. ($\Rightarrow$) Suppose the conclusion is not valid. Then there exist an $\alpha>0$ and a sequence $\{\varepsilon_n\}$ tending monotonically to zero as $n\to\infty$ such that $\|\Delta^{-1}(\varepsilon_n)(h(t;\varepsilon_n)-\lim^* f(t;\varepsilon_n))\|\geqq\alpha$ for $t=U_k(\varepsilon_n)$, $k=1,2,\cdots$, and $U_k(\varepsilon_n)\to\infty$ as $k\to\infty$.

From the definition of lim*, there exist numbers $V(\varepsilon_n)$ such that

$$\left\|\Delta^{-1}(\varepsilon_n)\left(f(t;\varepsilon_n)-\lim_{t\to\infty}{}^* f(t;\varepsilon_n)\right)\right\|<\alpha/2\quad\text{for all } t>V(\varepsilon_n).$$

Since $U_k(\varepsilon_n)\to\infty$ as $k\to\infty$, there are integers k_n such that $U_k(\varepsilon_n)>V(\varepsilon_n)$ for all $k\geqq k_n$. Hence

$$\|\Delta^{-1}(\varepsilon_n)(h(t;\varepsilon_n)-f(t;\varepsilon_n))\|>\alpha/2\quad\text{if } t=U_k(\varepsilon_n),\ k\geqq k_n.$$

Let ord $\delta\in R_\gamma$. Again using $U_k(\varepsilon_n)\to\infty$, we find integers $r_n\geqq k_n$ such that $\delta(\varepsilon_n)\leqq U_{r_n}(\varepsilon_n)$. Let θ be a continuous function of ε increasing monotonically without bound as $\varepsilon\to 0$, and satisfying $\theta(\varepsilon_n)=U_{r_n}(\varepsilon_n)$. Since $\lim_{\varepsilon\downarrow 0}(\theta(\varepsilon)/\delta(\varepsilon))\neq 0$, ord $\theta\in R_\gamma$.

It follows that for $n=1,2,\cdots$,

$$\|\Delta^{-1}(\varepsilon_n)(h(\theta(\varepsilon_n);\varepsilon_n)-f(\theta(\varepsilon_n);\varepsilon_n))\|\geqq\alpha/2,$$

so that $\lim_\theta\|\Delta^{-1}(h-f)\|\neq 0$ (it may not even exist). Thus h is not a Δ-approximation to f for ord $\theta\in R_\gamma$, a contradiction.

($\Leftarrow$) Without loss of generality, we may assume that $T_\alpha(\varepsilon)$ is monotone increasing as $\alpha\downarrow 0$. Let $\{\alpha_n\}$ be a positive sequence decreasing monotonically to zero, and let $\gamma(\varepsilon)$ be a continuous monotone function satisfying $\gamma(\alpha_n)=T_{\alpha_n}(\alpha_n)$.

Let ord $\theta\in R_\gamma$, and choose an arbitrary positive integer n and interval $[y_1,y_2]\subset(0,\infty)$. Then there exists an integer $N(y_1,y_2)$ such that $t\equiv t_\theta\theta(\varepsilon)>\gamma(\varepsilon)\geqq T_{\alpha n}(\alpha_n)$ if $t_\theta\in[y_1,y_2]$, $n\geqq N$, and $\varepsilon\leqq A_{\alpha_n}$. Consequently, for such t_θ, n, and ε,

$$\left\| \Delta^{-1}(\varepsilon)\left(h(t_\theta\theta(\varepsilon);\varepsilon) - \lim_{t\to\infty}{}^{*} f(t_\theta\theta(\varepsilon);\varepsilon)\right)\right\| < \alpha_n.$$

By definition of lim*, there exists a positive function $V(\varepsilon)$ such that $\|f(t;\varepsilon)-\lim^* f(t;\varepsilon)\| < \varepsilon\Delta(\varepsilon)$ if $t > V(\varepsilon)$. Thus $\operatorname{ord}\delta \in R_V$, $t_\delta \in [y_1, y_2]$ implies the existence of $A(y_1, y_2)$ such that $t \equiv t_\delta\delta(\varepsilon) > V(\varepsilon)$ if $\varepsilon < A$, and thus

$$\left\| \Delta^{-1}(\varepsilon)\left(f(t_\delta\delta(\varepsilon);\varepsilon) - \lim_{t\to\infty}{}^{*} f(t_\delta\delta(\varepsilon);\varepsilon)\right)\right\| < \varepsilon.$$

Hence, let R_ν be a right set contained in $R_V \cap R_\gamma$, and let $\operatorname{ord}\mu \in R_\nu$. Then

$$\|\Delta^{-1}(\varepsilon)(h(t_\mu\mu(\varepsilon);\varepsilon) - f(t_\mu\mu(\varepsilon);\varepsilon))\| < \alpha_n + \varepsilon \quad \text{if } t_\mu \in [y_1, y_2],$$

for all sufficiently large n and sufficiently small ε, i.e., $\lim_\mu \|\Delta^{-1}(f-h)\| = 0$ and h is a Δ-approximation to f in R_ν.

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