COMMUTANTS THAT DO NOT DILATE

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Abstract. The Lifting Theorem deals with dilation of the commutant of an operator $T_i$ on Hilbert space. In this note, counterexamples are given to generalizations of the theorem involving $N$ commuting operators $T_1, T_2, \ldots, T_N$.

In general terms, the Lifting Theorem for restricted shifts states that if $T$ is an operator commuting with the projection of the shift $S_1$ (on $H^2$) to one of its star-invariant subspaces, then $T$ may be dilated, without changing its norm, to an operator commuting with $S_1$. The theorem was first proved by Sarason [3], and has been extended by Sz.-Nagy and Foiaș to vector-valued $H^2$ spaces [4], and other, more general, situations [5].

To state the theorem more precisely, let us introduce some notation which at once suggests a different sort of generalization. Let $U^N$ denote the polydisk in $N$ complex variables $z_1, \ldots, z_N$. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $H^2(\mathcal{H})$ denote the $H^2$ space of $U^N$ based on $\mathcal{H}$. Thus an element of $H^2(\mathcal{H})$ has the form

$$f(z_1, \ldots, z_N) = \sum a_J z_{j_1}^1 z_{j_2}^2 \cdots z_{j_N}^N$$

where the sum is over $J = (j_1, \ldots, j_N) \in \mathbb{Z}_+^N$, where $a_J \in \mathcal{H}$, and where $\sum ||a_J||^2 < \infty$. Let $S_1, \ldots, S_N$ denote the shifts $(S_j f = z_j f)$ on $H^2(\mathcal{H})$, let $M$ denote a subspace of $H^2(\mathcal{H})$, invariant under $S_1, \ldots, S_N$, and define

$$T_j f = P_M z_j f, \quad f \in M^\perp = H^2(\mathcal{H}) \ominus M.$$

The above Lifting Theorem now states that, if $N=1$ and if $T$ commutes with $T_1$, then there is a dilation $S$ of $T$ which commutes with $S_1$ and which satisfies $||S|| = ||T||$.

The purpose of this note is to give examples of invariant subspaces $M$ in $H^2(\mathcal{H})$ and $H^3(\mathcal{H})$ ($=H^2_C(U^3)$, $C$ the complex numbers) and of bounded operators $T$ on $M^\perp$, commuting with the $T_j$, but having no (bounded) dilation commuting with the $S_j$. Such a $T$ always has an
unbounded dilation commuting with the $S_j$, as I proved in [1]. For an example involving a different dilation problem for commuting operators, see Parrott [2].

Our first example has to do with norms of dilations in $H^2(U^2)$ and generalizes an example I gave in [1].

**Example 1.** Let $M_n$ denote the invariant subspace of $H^2(U^2)$ generated by the homogeneous polynomials of degree $n$. If $p(z_1, z_2)$ is a homogeneous polynomial of degree $n$ and $T_p$ is the operator of multiplication by $p$ and projection on $M_{n+1}^\perp$, then $\|T_p\| = \|p\|_2$, but the minimal norm of a dilation of $T_p$ which commutes with $S_1$ and $S_2$ is $\|p\|_\infty$.

The first statement comes from the fact that $T_p$ has rank 1. In fact, $T_p1 = p$ and $T_p x = 0$ for $x \in M_{n+1}^\perp \cap \{1\}$.

To prove the second statement, note that an operator $T$ on $H^2(U^2)$ which commutes with $S_1$ and $S_2$ and which is a dilation of $T_p$ must consist of multiplication by a function $h$ of the form $p + f$, where $f \in M_{n+1}^\perp$. Pick $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha_1| = |\alpha_2|$ and $|p(\alpha)| = \|p\|_\infty$. If $h(\lambda) = p(\alpha) + f(\lambda \alpha_1, \lambda \alpha_2) / \lambda^n$, then $h$ is holomorphic in $|\lambda| < 1$ and $|h(0)| = \|p\|_\infty$. We have

$$\|T\| = \|p + f\|_\infty \geq \|p(\lambda \alpha) + f(\lambda \alpha)\|_\infty$$

where the last norm is the one variable $L^\infty$ norm. Further,

$$\|p(\lambda \alpha) + f(\lambda \alpha)\|_\infty = \|\lambda^n p(\alpha) + f(\lambda \alpha)\|_\infty = \|\lambda^n h(\lambda)\|_\infty$$

$$= \|h(\lambda)\|_\infty \geq |h(0)| = \|p\|_\infty.$$  

This completes Example 1.

**Example 2.** There is an invariant subspace $M$ of $H^2(U^2)$ and an operator $T$ on $M^\perp$ commuting with $T_1$ and $T_2$ which has no bounded dilation $S$ which commutes with $S_1$ and $S_2$.

Let $x_1, x_2, \cdots$ be an orthonormal basis of $\mathcal{H}$, and let $M$ consist of functions of the form $\sum a_{nm} z_1^n z_2^m$ where $a_{nm}$ lies in the span of $x_1, x_2, \cdots, x_{n+m-1}$. Let $Q_n$ denote the projection of $\mathcal{H}$ on the span of $x_n$, and let $p_1, p_2, \cdots$ be homogeneous polynomials of degrees 1, 2, $\cdots$, which satisfy

(1) $\sum_{n=1}^{\infty} \|p_n\|_2^2 < \infty$

and

(2) $\|p_n\|_\infty \to \infty$.

For $x \in M_1^\perp$,

$$x = \sum_{j,k=0}^\infty a_{jk} z_1^j z_2^k,$$

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2 This proof incorporates simplifications pointed out to me by H. Alexander.
where $a_{jk}$ is a linear combination of $x_{j+k}$, $x_{j+k+1}$, $\cdots$. Let $T'_n$ denote the operator on $M^\perp$ of multiplication by $p_nQ_n$ and projection on $M^\perp$. Thus

$$T'_n x = P_{M^n} p_n \sum_{j+k \leq n} z^n_{j+k} (Q_n a_{jk}),$$

and since $Q_n a_{jk} = \langle a_{jk}, x_n \rangle x_n$, we have $p_n z^n_{j+k} Q_n a_{jk} \in M$ if $j + k > 0$. It follows that

$$(3) \quad T'_n x = p_n Q_n a_{00} = \langle a_{00}, x_n \rangle p_n x_n.$$ 

Clearly $T'_n \cdot T_m = T'_m \cdot T_n$ if $n \neq m$ and so, by (1) and (3), $T = \sum_{n=0}^{\infty} T'_n$ exists in the strong operator topology and $T$ commutes with $T_1$ and $T_2$.

Now any (bounded) dilation $S$ of $T$ which commutes with $S_1$ and $S_2$ must have the form $Sf = p(z_1, z_2) f$ where $p$ is an analytic function in $U^2$ whose values are operators on $\mathcal{H}$ and $\|p(z_1, z_2)\| \leq K$, say. In addition, $S$ maps $M$ into $M$ and, if $f \in M^\perp$, $p(z_1, z_2) f = Tf + x$, where $x \in M$. If $x \in M$, if $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$, and if $(z_1, z_2) \in U^2$, we have

$$\langle x(z_1, z_2), x_n \rangle = \sum_{j+k = n+1}^{\infty} \langle a_{jk}, x_n \rangle z^n_{j+k}.$$ 

It follows that

$$K \geq |\langle Sx_n, x_n \rangle| = |\langle Tx_n, x_n \rangle + \langle x, x_n \rangle|$$

$$= |p_n(z_1, z_2) + \sum_{j+k = n+1}^{\infty} \langle a_{jk}, x_n \rangle z^n_{j+k}|$$

for $(z_1, z_2) \in U$ and for all $n$. This contradicts Example 1 and (2).

**Example 3.** There is an invariant subspace $M$ of $H^2(U^3) = H^2_2(U^3)$ and an operator $T$ on $M^\perp$ commuting with $T_1$, $T_2$ and $T_3$ which has no bounded dilation $S$ which commutes with $S_1$, $S_2$ and $S_3$.

Let $B(z)$ be a Blaschke product in one variable

$$B(z) = \prod_{n=1}^{\infty} \frac{-\bar{a}_n}{|a_n|} \frac{z - a_n}{1 - \bar{a}_n z},$$

whose zeros are distinct but otherwise unspecified for the moment. Let

$$B_m(z) = \prod_{n=m}^{\infty} \frac{-\bar{a}_n}{|a_n|} \frac{z - a_n}{1 - \bar{a}_n z},$$

and let $M_n$ be the invariant subspace of $H^2(U^3)$ generated by the homogeneous polynomials in $z_1$ and $z_2$ of degree $n$. Let $M$ denote the closure of the span of

$$B_1(z_3) M_1 \cup B_2(z_3) M_2 \cup \cdots.$$
Thus $M$ is the invariant subspace of $H^2(U^3)$ generated by all functions of the form $B_j(z_3)p(z_1, z_2)$ where $p$ is a homogeneous polynomial of degree $j$.

Again we choose homogeneous polynomials $p_0, p_1, \cdots$ in $z_1, z_2$ of degrees $0, 1, \cdots$ and satisfying (1) and (2). This time, $T'_n$ is the operator on $M$ of multiplication by $p_{n-1}B_n$ and projection on $M'$.

Clearly $T'_n f = 0$ (i.e. $p_{n-1}B_n f \in M$) if either $f \in M_1$ or $f(a_{n-1}) = 0$. Thus $T'_n$ has rank 1 and is zero on the orthogonal complement of the span of the function

$$F(z_1, z_2, z_3) = (1 - \tilde{a}_{n-1}z_2)^{-1}.$$ 

Furthermore,

$$T'_n F = p_{n-1}B_n(1 - \tilde{a}_{n-1}z_2)^{-1}$$

and

$$\|p_{n-1}B_n(1 - \tilde{a}_{n-1}z_2)^{-1}\| = \|p_{n-1}\| \|(1 - \tilde{a}_{n-1}z_2)^{-1}\|,$$

so that $\|T'_n\| \leq \|p_{n-1}\|$. In addition, (4) implies that the ranges of the $T'_n$ are orthogonal, so we may conclude that $T = \sum T'_n$ exists in the strong operator topology and commutes with $T_1, T_2$ and $T_3$. We claim there is no function $f \in M$ such that

$$\left\| \sum_{n=1}^{\infty} p_{n-1}(z_1, z_2)B_n(z_3) + f \right\|_\infty = K < \infty.$$ 

In fact, if $f \in M, f(z_1, z_2, a_n)$ has homogeneous degree at least $n+1$, so that, setting $z_3 = a_n$ in (5) gives

$$\|p_nB_{n+1}(a_n) + f(z_1, z_2, a_n)\|_\infty \leq K,$$

and $f(z_1, z_2, a_n) \in M_{n+1}$. If we now assume that $B$ is chosen so that $B_{n+1}(a_n)$ is bounded from 0 (i.e. if the sequence $\{a_n\}$ is interpolating) we have obtained a contradiction.

References


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