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COMMUTANTS THAT DO NOT DILATE

DOUGLAS N. CLARK[1]

ABSTRACT. The Lifting Theorem deals with dilation of the commutant of an operator T_1 on Hilbert space. In this note, counterexamples are given to generalizations of the theorem involving N commuting operators $T_1, T_2, \cdots, T_N$.

In general terms, the Lifting Theorem for restricted shifts states that if T is an operator commuting with the projection of the shift S_1 (on H^2) to one of its star-invariant subspaces, then T may be dilated, without changing its norm, to an operator commuting with S_1. The theorem was first proved by Sarason [3], and has been extended by Sz.-Nagy and Foiaş to vector-valued H^2 spaces [4], and other, more general, situations [5].

To state the theorem more precisely, let us introduce some notation which at once suggests a different sort of generalization. Let U^N denote the polydisk in N complex variables $z_1, \cdots, z_N$. Let $\mathscr{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $H^2_{\mathscr{H}}(U^N)$ denote the H^2 space of U^N based on $\mathscr{H}$. Thus an element of $H^2_{\mathscr{H}}(U^N)$ has the form

$$f(z_1, \cdots, z_N) = \sum a_J z_1^{j_1} z_2^{j_2} \cdots z_N^{j_N}$$

where the sum is over $J=(j_1, \cdots, j_N) \in Z_+^N$, where $a_J \in \mathscr{H}$, and where $\sum \|a_J\|^2 < \infty$. Let $S_1, \cdots, S_N$ denote the shifts ($S_j f = z_j f$) on $H^2_{\mathscr{H}}(U^N)$, let M denote a subspace of $H^2_{\mathscr{H}}(U^N)$, invariant under $S_1, \cdots, S_N$, and define

$$T_j f = P_{M^\perp} z_j f, \qquad f \in M^\perp = H^2_{\mathscr{H}}(U^N) \ominus M.$$

The above Lifting Theorem now states that, if $N=1$ and if T commutes with T_1, then there is a dilation S of T which commutes with S_1 and which satisfies $\|S\| = \|T\|$.

The purpose of this note is to give examples of invariant subspaces M in $H^2_{\mathscr{H}}(U^2)$ and $H^2(U^3)$ $(=H^2_C(U^3)$, C the complex numbers) and of bounded operators T on $M^\perp$, commuting with the T_j, but having no (bounded) dilation commuting with the S_j. Such a T always has an

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unbounded dilation commuting with the S_j, as I proved in [**1**]. For an example involving a different dilation problem for commuting operators, see Parrott [**2**].

Our first example has to do with norms of dilations in $H^2(U^2)$ and generalizes an example I gave in [**1**].

EXAMPLE 1. *Let M_n denote the invariant subspace of $H^2(U^2)$ generated by the homogeneous polynomials of degree n. If $p(z_1, z_2)$ is a homogeneous polynomial of degree n and T_p is the operator of multiplication by p and projection on $M_{n+1}^{\perp}$, then $\|T_p\|=\|p\|_2$, but the minimal norm of a dilation of T_p which commutes with S_1 and S_2 is $\|p\|_\infty$.*

The first statement comes from the fact that T_p has rank 1. In fact, $T_p 1=p$ and $T_p x=0$ for $x \in M_{n+1}^{\perp}\ominus\{1\}$.

To prove the second statement,[2] note that an operator T on $H^2(U^2)$ which commutes with S_1 and S_2 and which is a dilation of T_p must consist of multiplication by a function of the form $p+f$, where $f \in M_{n+1}$. Pick $\alpha=(\alpha_1, \alpha_2)$ with $|\alpha_1|=|\alpha_2|$ and $|p(\alpha)|=\|p\|_\infty$. If $h(\lambda)=p(\alpha)+f(\lambda\alpha_1, \lambda\alpha_2)/\lambda^n$, then h is holomorphic in $|\lambda|<1$ and $|h(0)|=\|p\|_\infty$. We have

$$\|T\| = \|p+f\|_\infty \geqq \|p(\lambda\alpha)+f(\lambda\alpha)\|_\infty$$

where the last norm is the one variable L^∞ norm. Further,

$$\begin{aligned}\|p(\lambda\alpha)+f(\lambda\alpha)\|_\infty &= \|\lambda^n p(\alpha)+f(\lambda\alpha)\|_\infty = \|\lambda^n h(\lambda)\|_\infty \\ &= \|h(\lambda)\|_\infty \geqq |h(0)| = \|p\|_\infty.\end{aligned}$$

This completes Example 1.

EXAMPLE 2. *There is an invariant subspace M of $H^2_{\mathscr{H}}(U^2)$ and an operator T on $M^{\perp}$ commuting with T_1 and T_2 which has no bounded dilation S which commutes with S_1 and S_2.*

Let $x_1, x_2, \cdots$ be an orthonormal basis of $\mathscr{H}$, and let M consist of functions of the form $\sum a_{nm} z_1^n z_2^m$ where a_{nm} lies in the span of $x_1, x_2, \cdots, x_{n+m-1}$. Let Q_n denote the projection of $\mathscr{H}$ on the span of x_n, and let $p_1, p_2, \cdots$ be homogeneous polynomials of degrees $1, 2, \cdots$, which satisfy

$$(1) \qquad \sum \|p_n\|_2^2 < \infty$$

and

$$(2) \qquad \|p_n\|_\infty \to \infty.$$

For $x \in M^{\perp}$,

$$x = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k,$$

[2] This proof incorporates simplifications pointed out to me by H. Alexander.

where a_{jk} is a linear combination of $x_{j+k}, x_{j+k+1}, \cdots$. Let T'_n denote the operator on $M^{\perp}$ of multiplication by $p_n Q_n$ and projection on $M^{\perp}$. Thus

$$T'_n x = P_{M^{\perp}} p_n \sum_{j+k \leqq n} z_1^j z_2^k (Q_n a_{jk}),$$

and since $Q_n a_{jk} = \langle a_{jk}, x_n \rangle x_n$, we have $p_n z_1^j z_2^k Q_n a_{jk} \in M$ if $j+k>0$. It follows that

$$T'_n x = p_n Q_n a_{00} = \langle a_{00}, x_n \rangle p_n x_n. \tag{3}$$

Clearly $T'_n x \perp T'_m x$ if $n \neq m$ and so, by (1) and (3), $T = \sum_{n=0}^{\infty} T'_n$ exists in the strong operator topology and T commutes with T_1 and T_2.

Now any (bounded) dilation S of T which commutes with S_1 and S_2 must have the form $Sf = p(z_1, z_2)f$ where p is an analytic function in U^2 whose values are operators on $\mathscr{H}$ and $\|p(z_1, z_2)\| \leqq K$, say. In addition, S maps M into M and, if $f \in M^{\perp}$, $p(z_1, z_2)f = Tf + x$, where $x \in M$. If $x \in M$, if $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathscr{H}$, and if $(z_1, z_2) \in U^2$, we have

$$\langle x(z_1, z_2), x_n \rangle = \sum_{j+k=n+1}^{\infty} \langle a_{jk}, x_n \rangle z_1^j z_2^k.$$

It follows that

$$\begin{aligned} K \geqq |\langle Sx_n, x_n \rangle| &= |\langle Tx_n, x_n \rangle + \langle x, x_n \rangle| \\ &= \left| p_n(z_1, z_2) + \sum_{j+k=n+1}^{\infty} \langle a_{jk}, x_n \rangle z_1^j z_2^k \right| \end{aligned}$$

for $(z_1, z_2) \in U$ and for all n. This contradicts Example 1 and (2).

EXAMPLE 3. *There is an invariant subspace M of $H^2(U^3) = H^2_C(U^3)$ and an operator T on $M^{\perp}$ commuting with T_1, T_2 and T_3 which has no bounded dilation S which commutes with S_1, S_2 and S_3.*

Let $B(z)$ be a Blaschke product in one variable

$$B(z) = \prod_{n=1}^{\infty} \frac{-\bar{a}_n}{|a_n|} \frac{z - a_n}{1 - \bar{a}_n z},$$

whose zeros are distinct but otherwise unspecified for the moment. Let

$$B_m(z) = \prod_{n=m}^{\infty} \frac{-\bar{a}_n}{|a_n|} \frac{z - a_n}{1 - \bar{a}_n z},$$

and let M_n be the invariant subspace of $H^2(U^3)$ generated by the homogeneous polynomials in z_1 and z_2 of degree n. Let M denote the closure of the span of

$$B_1(z_3)M_1 \cup B_2(z_3)M_2 \cup \cdots.$$

Thus M is the invariant subspace of $H^2(U^3)$ generated by all functions of the form $B_j(z_3)p(z_1, z_2)$ where p is a homogeneous polynomial of degree j.

Again we choose homogeneous polynomials $p_0, p_1, \cdots$ in z_1, z_2 of degrees $0, 1, \cdots$ and satisfying (1) and (2). This time, T'_n is the operator on $M^\perp$ of multiplication by $p_{n-1}B_n$ and projection on $M^\perp$.

Clearly $T'_n f=0$ (i.e. $p_{n-1}B_n f \in M$) if either $f \in M_1$ or $f(a_{n-1})=0$. Thus T'_n has rank 1 and is zero on the orthogonal complement of the span of the function

$$F(z_1, z_2, z_3) = (1 - \bar{a}_{n-1}z_3)^{-1}.$$

Furthermore,

$$T'_n F = p_{n-1}B_n(1 - \bar{a}_{n-1}z_3)^{-1} \tag{4}$$

and

$$\|p_{n-1}B_n(1 - \bar{a}_{n-1}z_3)^{-1}\| = \|p_{n-1}\| \, \|(1 - \bar{a}_{n-1}z_3)^{-1}\|,$$

so that $\|T'_n\| \leqq \|p_{n-1}\|$. In addition, (4) implies that the ranges of the T'_n are orthogonal, so we may conclude that $T=\sum_n T'_n$ exists in the strong operator topology and commutes with T_1, T_2 and T_3. We claim there is no function $f \in M$ such that

$$\left\| \sum_{n=1}^{\infty} p_{n-1}(z_1, z_2)B_n(z_3) + f \right\|_\infty = K < \infty. \tag{5}$$

In fact, if $f \in M$, $f(z_1, z_2, a_n)$ has homogeneous degree at least $n+1$, so that, setting $z_3=a_n$ in (5) gives

$$\|p_n B_{n+1}(a_n) + f(z_1, z_2, a_n)\|_\infty \leqq K,$$

and $f(z_1, z_2, a_n) \in M_{n+1}$. If we now assume that B is chosen so that $B_{n+1}(a_n)$ is bounded from 0 (i.e. if the sequence $\{a_n\}$ is interpolating) we have obtained a contradiction.

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Department of Mathematics, University of California, Los Angeles, California 90024