PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

SUMS OF DISTANCES BETWEEN POINTS ON A SPHERE

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ABSTRACT. An upper bound for the sum of the λth powers of all distances determined by N points on a unit sphere is given for $0 \leqq \lambda \leqq 1$.

Let $p_1, \cdots, p_N$ be points on the unit sphere U^m of E^m, the m-dimensional Euclidean space. Let

$$S = S(N, m, \lambda) = \max \sum_{i<j} |p_i - p_j|^\lambda$$

where the maximum is taken over all possible $p_1, \cdots, p_N$. The function S has been studied in [1]–[5]. Let $\sigma(U^m)$ denote the surface area of U^m. Björck has shown in [3] that $S < c(m, \lambda)N^2$ where $c(m, \lambda) = \frac{1}{2}\sigma(U^m)^{-1} \int |p_0 - p|^\lambda \, d\sigma(p)$; here $\int \cdots d\sigma(p)$ denotes a surface integral over U^m and p_0 is an arbitrary point on U^m. For example, $c(3, 1) = \frac{2}{3}$. In [1] Alexander used Haar integrals to prove

$$S(N, 3, 1) < \tfrac{2}{3}N^2 - \tfrac{1}{2}. \tag{1}$$

By the same technique we will show (see comment after Theorem 2) that for $0 \leqq \lambda \leqq 1$,

$$S(N, m, \lambda) \leqq c(m, \lambda)N^2 - \tfrac{1}{2}c(m, \lambda) \tag{2}$$

is true at least "half of the time"; more precisely, for any positive integer n the inequality (2) is true either for $N = n$ or for $N = n + 1$. Our proof is based on a new generalization of the triangle inequality, Theorem 1, and an imbedding theorem of I. J. Schoenberg [6] or [7, p. 527]. Note that when $m = 3$ and $\lambda = 1$ our result is weaker than Alexander's.

LEMMA 1. *Let $u \geqq v + 2 \geqq 3$ and let H denote the hexagonal region $0 \leqq x \leqq u$, $0 \leqq y \leqq v$, $1 \leqq x + y \leqq u + v - 1$. Then for (x, y) in H we have*

$$g(x, y) = \frac{x(u - x) + y(v - y)}{x(v - y) + y(u - x)} \leqq \frac{u - 1}{v}.$$

Received by the editors December 29, 1971.
AMS 1970 *subject classifications*. Primary 52A25, 52A40.
[1] Research supported in part by the National Science Foundation under contract GP-12363.

PROOF. Let s denote the substitution which takes x and y to $u-x$ and $v-y$ respectively. The function $g(x, y)$ is invariant under s, so it suffices to prove the inequality for the three critical points $(\frac{1}{4}(u+v), \frac{1}{4}(u+v))$, $(\frac{1}{2}(u+v), \frac{1}{2}(u+v))$, $(\frac{1}{2}u, \frac{1}{2}v)$, and the three boundary lines $x+y=1$, $y=0$ and $x=u$. The straightforward details are omitted.

Now let S_1 be a finite set of u points $p_1, \cdots, p_u$ in E^m and similarly let S_2 be a set of v such points $q_1, \cdots, q_v$. Let $\sum_1 = \sum_{i<j} |p_i - p_j|$, $\sum_2 = \sum_{i<j} |q_i - q_j|$, and $\sum_{12} = \sum_{i,j} |p_i - q_j|$.

THEOREM 1. *Let $u \geqq v$. Define*

$$f(u, v) = 1, \qquad u = v,$$
$$= (u-1)/v, \qquad u > v.$$

Then

$$\sum_1 + \sum_2 \leqq f(u, v) \sum_{12}. \tag{3}$$

Moreover the constant $f(u, v)$ is best possible.

PROOF. For any two points $p_1, p_2 \in E^m$,

$$|p_1 - p_2| = a_m \int |p_1 \cdot p - p_2 \cdot p| \, d\sigma(p)$$

where a_m depends only on the dimension m. Hence it suffices to prove (3) in the case where the p_i and q_j are real numbers. Let J be any interval on the real axis which does not contain any point p_i or q_j. Let x be the number of p_i to the left of J and y the number of q_j to the left of J. It suffices to show $L \leqq f(u, v)R$ where L is the number of times J is counted on the left of (3) and R is the number of times it is counted on the right, i.e. that

$$x(u-x) + y(v-y) \leqq f(u, v)\{x(v-y) + y(u-x)\}.$$

For $v \leqq u \leqq v+1$ this follows from $(x-y)(u-v) \leqq (x-y)^2$ and for $v+2 \leqq u$ it follows from Lemma 1. Equality holds in (3) if (*not* only if) $p_2 = p_3 = \cdots = p_u = q_1 = \cdots = q_v$ and p_1 is arbitrary.

THEOREM 2. *For $0 \leqq \lambda \leqq 1$,*

$$S(u, m, \lambda) + S(v, m, \lambda) \leqq 2c(m, \lambda)uvf(u, v). \tag{4}$$

PROOF. Fix λ. Given any t points $p_1, \cdots, p_t$ in E^m, we can find an $m' \geqq m$ and t points $p'_1, \cdots, p'_t$ in $E^{m'}$ such that $|p'_i - p'_j| = |p_i - p_j|^\lambda$ for $1 \leqq i, j \leqq t$. This result is due to I. J. Schoenberg [7, p. 527]. In the following τ, τ_1, τ_2 shall denote elements of the special orthogonal group $SO(m)$ acting on U^m. The symbol $\int \cdots d\tau$ shall denote a Haar integral over that group; we normalize the measure so that $\int_{SO(m)} d\tau = 1$. Choose $p_1, \cdots, p_u$

and $q_1, \cdots, q_v$ on U^m so that

$$(5)\quad S(u, m, \lambda) = \sum_{i<j} |p_i - p_j|^\lambda \quad \text{and} \quad S(v, m, \lambda) = \sum_{i<j} |q_i - q_j|^\lambda.$$

Define

$$I = \int \sum_{i,j} |p_i - \tau_1 q_j|^\lambda \, d\tau_1.$$

Clearly

$$(6)\quad I = uv \int |p - \tau_1 \cdot p|^\lambda \, d\tau_1 = uv\sigma(U^m)^{-1} \int |p_0 - p|^\lambda \, d\sigma(p) = 2c(m, \lambda)uv$$

On the other hand, for any p_0 on $U^{m'}$,

$$|p_i - \tau \cdot q_j|^\lambda = |p_i' - (\tau_1 q_j)'| = b_{m'} \int |(p_i' - (\tau_1 \cdot q_j)') \cdot \tau_2 p_0| \, d\tau_2$$

where $b_{m'}$ depends only on the dimension m'. Hence by Theorem 1,

$$(7)\quad \begin{aligned} I &= \iint b_{m'} \sum_{i,j} |p_i' \cdot \tau_2 p_0 - (\tau_1 q_j)' \cdot \tau_2 p_0| \, d\tau_2 \, d\tau_1 \\ &\geqq f^{-1} \iint \Big(\sum_{i<j} b_{m'} |(p_i' - p_j') \cdot \tau_2 p_0| \\ &\qquad + \sum_{i<j} b_{m'} |\{(\tau_1 q_i)' - (\tau_1 q_j)'\} \cdot \tau_2 p_0| \Big) \, d\tau_2 \, d\tau_1 \\ &= f^{-1} \int \Big(\sum_{i<j} |p_i - p_j|^\lambda + \sum_{i<j} |q_i - q_j|^\lambda \Big) \, d\tau_1. \end{aligned}$$

The result follows from (5), (6), and (7).

To obtain (2) set $v = u - 1$ in (4) and note that $cu^2 - \frac{1}{2}c + c(u-1)^2 - \frac{1}{2}c = 2cu(u-1)$.

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