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CLOSED IDEALS IN $C(X)$

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ABSTRACT. The characterization of uniformly closed ideals in $C(X)$, for X compact Hausdorff, is well known. In this note, we extend this characterization to an arbitrary completely regular Hausdorff X and derive some corollaries.

1. **Preliminaries.** We shall assume that the reader is familiar with the terminology and basic results of the Gillman and Jerison text **[GJ]**. Thus, $C(X)$ will denote the algebra of all continuous real-valued functions defined on the space X. The class of algebras $C(X)$ is unaltered if we restrict attention to completely regular Hausdorff spaces X, and therefore *X will always denote a completely regular Hausdorff space in the sequel.*

Each continuous $f: X \to \boldsymbol{R}$ admits a unique continuous extension $f^*: \beta X \to \gamma \boldsymbol{R}$ where βX denotes the Stone-Čech compactification of X and $\gamma \boldsymbol{R}$ denotes the extended reals (the two-point compactification of the reals $\boldsymbol{R}$). For $I \subseteq C(X)$, we write $I^* = \{f^* : f \in I\}$. We shall expand the zero-set notation $Z(f)$ and $Z[I]$ to include extended real-valued functions.

We may define a metric ρ on $C(X)$ by the formula

$$\rho(f, g) = \sup\{|f(x) - g(x)| \wedge 1 : x \in X\}.$$

This metric is complete, and $C(X)$ becomes a topological vector space, but in general not a topological ring, in the metric topology. This topology is called the *uniform topology* (or u-*topology*), and the reader is referred to **[H]** for further details. If $I \subseteq C(X)$, then $\bar{I}$ will denote the uniform closure of I. In the remainder of this note, all topological properties of $C(X)$ will refer to the uniform topology.

By "ideal", we shall mean "proper ring ideal".

2. **Closures of ideals.** If I is an ideal in $C(X)$, then its closure $\bar{I}$ is easily seen to be a proper closed vector sublattice of $C(X)$. However, $\bar{I}$ need not be an ideal; there may exist $f \in \bar{I}$ and $g \in C(X)$ such that $fg \notin \bar{I}$.

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In fact, the next result guarantees the existence of such an I for any nonpseudocompact space X.

2.1 THEOREM. *The following conditions are mutually equivalent.*

(a) *X is pseudocompact.*

(b) *The closure of any ideal in $C(X)$ is an ideal.*

(c) *Each ideal in $C(X)$ is contained in a closed ideal.*

PROOF. (a) implies (b). If X is pseudocompact, then $C(X)=C(\beta X)$ is a topological algebra under the uniform (norm) topology.

(b) implies (c). Clear.

(c) implies (a). Suppose that X is not pseudocompact; thus, there exists an unbounded $f\in C(X)$ which we may assume to be strictly positive. Define

$$F_n=\{x\in X:f(x)\geqq n\},\qquad n=1,2,3,\cdots,$$

$$I=\{g\in C(X):F_n\subseteq Z(g)\text{ for some }n\}.$$

Then I is an ideal in $C(X)$. Suppose that I were contained in the closed ideal J; we shall show that $1/f\in J$ in contradiction to the definition of J. Let $\varepsilon>0$ (ε real) be given, and choose a positive integer n such that $1/n<\varepsilon$. Define $g=(1/f-1/n)\vee 0$. Then $F_n\subseteq Z(g)$, so that $g\in I\subseteq J$, while $0<(1/f)-g$ $=1/n\wedge 1/f\leqq 1/n<\varepsilon$. Hence $1/f\in J$. □

2.2 EXAMPLE. A *particular* ideal can be contained in a closed ideal without its closure being an ideal. Let X be nonpseudocompact, and let I be the ideal constructed in 2.1. Let Y be the disjoint union of X and the one-point space $\{p\}$, and define

$$J=\{f\in C(Y):f|X\in I,f(p)=0\},$$

$$K=\{f\in C(Y):f|X\in \bar{I},f(p)=0\},$$

$$M=\{f\in C(Y):f(p)=0\}.$$

Then J is an ideal in $C(Y)$ which is contained in the closed ideal M, while $\bar{J}=K$ is not an ideal. □

Even though the closure of an ideal I need not be an ideal, we can still give an explicit formula for $\bar{I}$ (cf. **[GJ**, 4O]).

2.3 THEOREM. *If I is an ideal in $C(X)$, then*

$$\bar{I}=\{f\in C(X):Z(f^*)\supseteq \bigcap Z[I^*]\}.$$

PROOF. Let I be an ideal in $C(X)$, and define $\Delta=\bigcap Z[I^*]$ and $J=\{f\in C(X):Z(f^*)\supseteq\Delta\}$. Clearly J is closed and $I\subseteq J$. It suffices to show that $J\subseteq\bar{I}$. So let $f\in J$ and $\varepsilon>0$ (ε real) be given. Letting

$$g=[(f-\varepsilon)\vee 0]+[(f+\varepsilon)\wedge 0],$$

we have $|f-g|\leqq\varepsilon$ and $Z(g^*)\supseteq f^{*\leftarrow}((-\varepsilon, \varepsilon))$, a neighborhood of Δ. We must show that $g \in I$. By compactness of βX, there exist $h_1, h_2, \cdots, h_n \in I$ such that $Z(g^*)$ is a neighborhood of $\bigcap_{i=1}^n Z(h_i^*)$. Defining $h=h_1^2+h_2^2+\cdots+h_n^2$, we have $h \in I$ and $Z(h^*)=\bigcap_{i=1}^n Z(h_i^*)$. If we let

$$\begin{aligned} k(x) &= g(x)/h(x) \quad \text{for } x \in X \sim Z(g), \\ &= 0 \qquad\qquad \text{for } x \in Z(g), \end{aligned}$$

then $k \in C(X)$ and $g=kh \in I$. □

2.4 Corollary. *An ideal I in $C(X)$ is closed if and only if*

$$I = \{f \in C(X):Z(f^*) \supseteq \bigcap Z[I^*]\}.$$

3. **Ideal sets.** We have shown that a closed ideal in $C(X)$ consists of all functions f whose extensions f^* vanish on some fixed nonvoid compact set. Let us now consider the problem in reverse. That is, let Δ be some nonvoid compact subset of βX, and form the set $I=\{f \in C(X):Z(f^*)\supseteq \Delta\}$. Then I is a closed vector sublattice of $C(X)$ but need not be an ideal. For example, let $X=N$, the discrete space of positive integers, and $\Delta=\{p\}$ where $p \in \beta N \sim N$. Then I contains the unit f, where $f(n)=1/n$, even though $I\neq C(X)$. We shall call Δ an *ideal set* if I is an ideal.

We now give a topological characterization of ideal sets, but first we need a definition. We shall say that a subset S of βX is *far from X* if there exists a zero-set Z of βX such that $S\subseteq Z\subseteq \beta X\sim X$; otherwise S is *close to X*. Thus, X is realcompact if and only if each singleton subset of $\beta X\sim X$ is far from X [**GJ**, 8.8], and X is Lindelöf if and only if each compact subset of $\beta X\sim X$ is far from X [**S**]. Note that, by [**GJ**, 7D(1)],

$$\mathrm{cl}_{\beta X} Z(f) = \{p \in \beta X: (fg)^*(p) = 0 \text{ for all } g \in C(X)\}$$

for $f \in C(X)$.

3.1 Theorem. *The following conditions are mutually equivalent for any nonvoid compact subset Δ of βX.*

(a) *Δ is an ideal set.*

(b) *$\Delta=\bigcap Z[I^*]$ for some closed ideal I in $C(X)$.*

(c) *$Z(f^*)\supseteq \Delta$ implies $\mathrm{cl}_{\beta X} Z(f)\supseteq \Delta$ for all $f \in C(X)$.*

(d) *If S is far from X, then $\mathrm{cl}_{\beta X}(\Delta\sim S)=\Delta$.*

Proof. (a) implies (b). Let $I=\{f \in C(X):Z(f^*)\supseteq \Delta\}$.

(b) implies (c). Suppose that $\Delta=\bigcap Z[I^*]$ for some closed ideal I; then by 2.4, $I=\{f \in C(X):Z(f^*)\supseteq\Delta\}$. Let $f \in C(X)$ with $Z(f^*)\supseteq\Delta$. Then $f \in I$, whence $fg \in I$ for all $g \in C(X)$. But then

$$\mathrm{cl}_{\beta X} Z(f) = \bigcap \{Z((fg)^*):g \in C(X)\} \supseteq \Delta.$$

(c) implies (d). Suppose that S is far from X, but there exists $p \in \Delta$ with $p \notin \mathrm{cl}_{\beta X}(\Delta \sim S)$. Then there exist $h, k \in C^*(X)$ such that $S \subseteq Z(h^*) \subseteq \beta X \sim X$, $p \notin Z(k^*)$ and $\Delta \sim S \subseteq Z(k^*)$. Let $f = hk$ and $g = 1/h$. Then $Z(f^*) = Z(h^*) \cup Z(k^*) \supseteq \Delta$, but $p \notin Z(k^*) = Z((fg)^*)$ so that $p \notin \mathrm{cl}_{\beta X} Z(f)$.

(d) implies (a). Suppose that Δ is not an ideal set. Thus, if we let $I = \{f \in C(X): Z(f^*) \supseteq \Delta\}$, then there exist $f, g \in C(X)$ such that $f \in I$ and $fg \notin I$. So $Z(f^*) \supseteq \Delta$ and for some $p \in \Delta$, $p \notin Z((fg)^*)$. Let Z be a zero-set neighborhood of p in βX such that $Z \cap Z((fg)^*) = \varnothing$; then $S = \Delta \cap Z$ is far from X, since $S \subseteq Z(f^*) \cap Z \subseteq \beta X \sim X$. But Z is a neighborhood of p which does not meet $\Delta \sim S$, so $\mathrm{cl}_{\beta X}(\Delta \sim S) \neq \Delta$. □

It follows from 3.1 and 2.4 that every closed ideal is a z-ideal and therefore is absolutely convex (i.e. is an l-ideal; cf. [**P**, 3.7]).

It is clear from 3.1 that an ideal set must be close to X. The converse, however, does not hold. For example, let $X = N$ and $\Delta = \{1, p\}$ where $p \in \beta N \sim N$. Then Δ is close to X, but is not an ideal set. We do have the following partial converse.

3.2 Lemma. *Any compact subset of βX which is close to X contains an ideal set.*

Proof. Let K be a compact subset of βX which is close to X, and define $\Delta = \bigcap \{\mathrm{cl}_{\beta X} Z(h) : Z(h^*) \supseteq K\}$, a nonvoid compact subset of K. We shall use 3.1(c) to show that Δ is an ideal set. Thus, suppose that $Z(f^*) \supseteq \Delta$ for some $f \in C(X)$. By the definition of Δ, for each $p \in \beta X \sim Z(f^*)$, there exists $h \in C(X)$ such that $Z(h^*) \supseteq K$ and $p \notin \mathrm{cl}_{\beta X} Z(h)$. Since $\beta X \sim Z(f^*)$ is an F_σ in βX, it is Lindelöf, and therefore we can find $g_1, g_2, g_3, \cdots \in C(X)$ such that $Z(g_n^*) \supseteq K$ for each n, and $\bigcap_{n=1}^\infty \mathrm{cl}_{\beta X} Z(g_n) \subseteq Z(f^*)$. Defining $g = \sum_{n=1}^\infty (1/2^n)(|g_n| \wedge 1) \in C(X)$, we have $Z(g) = \bigcap_{n=1}^\infty Z(g_n) \subseteq Z(f)$ and $Z(g^*) = \bigcap_{n=1}^\infty Z(g_n^*) \supseteq K$. It follows that $\mathrm{cl}_{\beta X} Z(f) \supseteq \mathrm{cl}_{\beta X} Z(g) \supseteq \Delta$. □

4. **Some corollaries.** We now consider some consequences of 3.1 and 3.2. The first result follows also from [**P**, 2.6], where a more algebraic proof is given.

4.1 Corollary. *If the ideal I in $C(X)$ is contained in a unique maximal ideal (e.g., if I is prime), then I is closed if and only if I is a real ideal.*

Proof. Clearly a real ideal is closed [**GJ**, 8.4]. Suppose that I is a closed ideal which is contained in the unique maximal ideal M^p for some $p \in \beta X$. Then $O^p \subseteq I$ [**GJ**, 7.13], from which it follows that $\bigcap Z[I^*] = \{p\}$. Thus, $\{p\}$ is an ideal set, and it follows from 3.1 that, if $q \in \beta X \sim vX$, then $\mathrm{cl}_{\beta X}(\{p\} \sim \{q\}) = \{p\}$—i.e., $q \neq p$. Hence $p \in vX$ and $I = M^p$, a real ideal. □

It is clear that, for any nonvoid $E \subseteq X$, the set $\mathrm{cl}_{\beta X} E$ is an ideal set.

The next result characterizes those X for which all ideal sets are of this form. It can also be deduced from [**P**, 3.3 and 3.4].

4.2 Corollary. *The following conditions are mutually equivalent.*

(a) *X is Lindelöf.*

(b) *Every ideal set in βX is of the form* $\text{cl}_{\beta X} E$ *for some* $E \subseteq X$.

(c) *Every ideal set in βX meets X.*

(d) *Every closed ideal in $C(X)$ is an intersection of fixed maximal ideals.*

(e) *Every closed ideal in $C(X)$ is fixed.*

Proof. (a) implies (b). Assume (a), let Δ be an ideal set in βX, and suppose that $p \in \beta X \sim \text{cl}_{\beta X}(\Delta \cap X)$. Let F be a closed neighborhood of p such that $F \cap (\Delta \cap X) = \varnothing$. Then $S = F \cap \Delta$ is a compact subset of $\beta X \sim X$, and hence by 3.1(d), $p \notin \text{cl}_{\beta X}(\Delta \sim S) = \Delta$. Hence $\Delta = \text{cl}_{\beta X}(\Delta \cap X)$, and (b) holds.

(b) implies (c). Obvious.

(c) implies (a). Assume that (a) is false, so there exists a compact subset K of $\beta X \sim X$ which is close to X. By 3.2, K contains an ideal set Δ, so that (c) is false.

(b) if and only if (d). This follows easily from the fact that, for any $E \subseteq X, f \in \bigcap \{M^p : p \in E\}$ if and only if $\text{cl}_{\beta X} E \subseteq Z(f^*)$.

(c) if and only if (e). A closed ideal $\{f \in C(X) : Z(f^*) \supseteq \Delta\}$ is fixed if and only if Δ meets X. □

In 4.1 we showed that an ideal in $C(X)$ is a closed maximal ideal if and only if it is real. Clearly every closed maximal ideal is a maximal closed ideal. We conclude by proving the converse.

4.3 Corollary. *An ideal in $C(X)$ is a closed maximal ideal if and only if it is a maximal closed ideal.*

Proof. It suffices to show that every maximal closed ideal is a maximal ideal. Thus, suppose that I is a maximal closed ideal in $C(X)$, and let $\Delta = \bigcap Z[I^*]$. If Δ is not a singleton set, say $q_1, q_2 \in \Delta$ with $q_1 \neq q_2$, then there exist compact sets K_1 and K_2 such that $K_1 \cup K_2 = \Delta$, $q_1 \notin K_1$ and $q_2 \notin K_2$. At least one of K_1 and K_2, say K_1, must be close to X. By 3.2, K_1 contains an ideal set Δ_1. But since $q_1 \in \Delta \sim \Delta_1$, the closed ideal $J = \{f \in C(X) : Z(f^*) \supseteq \Delta_1\}$ is strictly bigger than I, contradicting the maximality of I. Therefore, we must have $\Delta = \{p\}$ for some $p \in \beta X$. But then $p \in vX$, and $I = M^p$, a maximal ideal. □

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