CLOSED IDEALS IN $C(X)$

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Abstract. The characterization of uniformly closed ideals in $C(X)$, for $X$ compact Hausdorff, is well known. In this note, we extend this characterization to an arbitrary completely regular Hausdorff $X$ and derive some corollaries.

1. Preliminaries. We shall assume that the reader is familiar with the terminology and basic results of the Gillman and Jerison text [GJ]. Thus, $C(X)$ will denote the algebra of all continuous real-valued functions defined on the space $X$. The class of algebras $C(X)$ is unaltered if we restrict attention to completely regular Hausdorff spaces $X$, and therefore $X$ will always denote a completely regular Hausdorff space in the sequel.

Each continuous $f: X \to \mathbb{R}$ admits a unique continuous extension $f^*: \beta X \to \gamma \mathbb{R}$ where $\beta X$ denotes the Stone-Čech compactification of $X$ and $\gamma \mathbb{R}$ denotes the extended reals (the two-point compactification of the reals $\mathbb{R}$). For $I \subseteq C(X)$, we write $I^* = \{f^*: f \in I\}$. We shall expand the zero-set notation $Z(f)$ and $Z[I]$ to include extended real-valued functions.

We may define a metric $\rho$ on $C(X)$ by the formula

$$\rho(f, g) = \sup\{|f(x) - g(x)| : x \in X\}.$$ 

This metric is complete, and $C(X)$ becomes a topological vector space, but in general not a topological ring, in the metric topology. This topology is called the uniform topology (or u-topology), and the reader is referred to [H] for further details. If $I \subseteq C(X)$, then $\overline{I}$ will denote the uniform closure of $I$. In the remainder of this note, all topological properties of $C(X)$ will refer to the uniform topology.

By "ideal", we shall mean "proper ring ideal".

2. Closures of ideals. If $I$ is an ideal in $C(X)$, then its closure $\overline{I}$ is easily seen to be a proper closed vector sublattice of $C(X)$. However, $\overline{I}$ need not be an ideal; there may exist $f \in I$ and $g \in C(X)$ such that $fg \notin I$.

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In fact, the next result guarantees the existence of such an \( I \) for any non-pseudocompact space \( X \).

**2.1 Theorem.** The following conditions are mutually equivalent.

(a) \( X \) is pseudocompact.

(b) The closure of any ideal in \( C(X) \) is an ideal.

(c) Each ideal in \( C(X) \) is contained in a closed ideal.

**Proof.** (a) implies (b). If \( X \) is pseudocompact, then \( C(X) = C(\beta X) \) is a topological algebra under the uniform (norm) topology.

(b) implies (c). Clear.

(c) implies (a). Suppose that \( X \) is not pseudocompact; thus, there exists an unbounded \( f \in C(X) \) which we may assume to be strictly positive. Define

\[
F_n = \{ x \in X : f(x) \geq n \}, \quad n = 1, 2, 3, \ldots,
\]

\[
I = \{ g \in C(X) : F_n \subseteq Z(g) \text{ for some } n \}.
\]

Then \( I \) is an ideal in \( C(X) \). Suppose that \( I \) were contained in the closed ideal \( J \); we shall show that \( 1/f \in J \) in contradiction to the definition of \( J \).

Let \( \epsilon > 0 \) (\( \epsilon \) real) be given, and choose a positive integer \( n \) such that \( 1/n < \epsilon \). Define \( g = (1/f - 1/n) \vee 0 \). Then \( F_n \subseteq Z(g) \), so that \( g \in I \subseteq J \), while \( 0 < (1/f) - g = 1/n \wedge 1/n \leq 1/n < \epsilon \). Hence \( 1/f \in J \). \( \square \)

**2.2 Example.** A particular ideal can be contained in a closed ideal without its closure being an ideal. Let \( X \) be nonpseudocompact, and let \( I \) be the ideal constructed in 2.1. Let \( Y \) be the disjoint union of \( X \) and the one-point space \( \{ p \} \), and define

\[
J = \{ f \in C(Y) : f|X \in I, f(p) = 0 \},
\]

\[
K = \{ f \in C(Y) : f|X \in I, f(p) = 0 \},
\]

\[
M = \{ f \in C(Y) : f(p) = 0 \}.
\]

Then \( J \) is an ideal in \( C(Y) \) which is contained in the closed ideal \( M \), while \( J = K \) is not an ideal. \( \square \)

Even though the closure of an ideal \( I \) need not be an ideal, we can still give an explicit formula for \( I \) (cf. [GJ, 40]).

**2.3 Theorem.** If \( I \) is an ideal in \( C(X) \), then

\[
I = \{ f \in C(X) : Z(f^*) \supseteq \bigcap Z[I^*] \}.
\]

**Proof.** Let \( I \) be an ideal in \( C(X) \), and define \( \Delta = \bigcap Z[I^*] \) and \( J = \{ f \in C(X) : Z(f^*) \supseteq \Delta \} \). Clearly \( J \) is closed and \( I \subseteq J \). It suffices to show that \( J \subseteq I \). So let \( f \in J \) and \( \epsilon > 0 \) (\( \epsilon \) real) be given. Letting

\[
g = [(f - \epsilon) \vee 0] + [(f + \epsilon) \wedge 0],
\]
we have \(|f-g| \leq \varepsilon\) and \(Z(g^*) \supseteq f^* + ((-\varepsilon, \varepsilon))\), a neighborhood of \(\Delta\). We must show that \(g \in I\). By compactness of \(\beta X\), there exist \(h_1, h_2, \ldots, h_n \in I\) such that \(Z(g^*)\) is a neighborhood of \(\bigcap_{i=1}^{n} Z(h_i^*)\). Defining \(h = h_1^2 + h_2^2 + \cdots + h_n^2\), we have \(h \in I\) and \(Z(h^*) = \bigcap_{i=1}^{n} Z(h_i^*)\). If we let

\[
k(x) = g(x)/h(x) \quad \text{for } x \in X \sim Z(g),
\]

\[
= 0 \quad \text{for } x \in Z(g),
\]

then \(k \in C(X)\) and \(g = kh \in I\).

\[\square\]

2.4 Corollary. An ideal \(I\) in \(C(X)\) is closed if and only if

\[I = \{f \in C(X) : Z(f^*) \supseteq \bigcap Z[I^*]\}.
\]

3. Ideal sets. We have shown that a closed ideal in \(C(X)\) consists of all functions \(f\) whose extensions \(f^*\) vanish on some fixed nonvoid compact set. Let us now consider the problem in reverse. That is, let \(\Delta\) be some nonvoid compact subset of \(\beta X\), and form the set \(I = \{f \in C(X) : Z(f^*) \supseteq \Delta\}\). Then \(I\) is a closed vector sublattice of \(C(X)\) but need not be an ideal. For example, let \(X = \mathbb{N}\), the discrete space of positive integers, and \(\Delta = \{p\}\) where \(p \in \beta \mathbb{N} \sim \mathbb{N}\). Then \(I\) contains the unit \(f\), where \(f(n) = 1/n\), even though \(I \neq C(X)\). We shall call \(\Delta\) an ideal set if \(I\) is an ideal.

We now give a topological characterization of ideal sets, but first we need a definition. We shall say that a subset \(S\) of \(\beta X\) is far from \(X\) if there exists a zero-set \(Z\) of \(\beta X\) such that \(S \subseteq Z \subseteq \beta X \sim X\); otherwise \(S\) is close to \(X\). Thus, \(X\) is realcompact if and only if each singleton subset of \(\beta X \sim X\) is far from \(X\) \([GJ, 8.8]\), and \(X\) is Lindelöf if and only if each compact subset of \(\beta X \sim X\) is far from \(X\) \([S]\). Note that, by \([GJ, 7D(1)]\),

\[
\operatorname{cl}_{\beta X} Z(f) = \{p \in \beta X : (fg)^*(p) = 0 \text{ for all } g \in C(X)\}
\]

for \(f \in C(X)\).

3.1 Theorem. The following conditions are mutually equivalent for any nonvoid compact subset \(\Delta\) of \(\beta X\).

(a) \(\Delta\) is an ideal set.

(b) \(\Delta = \bigcap Z[I^*]\) for some closed ideal \(I\) in \(C(X)\).

(c) \(Z(f^*) \supseteq \Delta\) implies \(\operatorname{cl}_{\beta X} Z(f) \supseteq \Delta\) for all \(f \in C(X)\).

(d) If \(S\) is far from \(X\), then \(\operatorname{cl}_{\beta X} (\Delta \sim S) = \Delta\).

Proof. (a) implies (b). Let \(I = \{f \in C(X) : Z(f^*) \supseteq \Delta\}\).

(b) implies (c). Suppose that \(\Delta = \bigcap Z[I^*]\) for some closed ideal \(I\); then by 2.4, \(I = \{f \in C(X) : Z(f^*) \supseteq \Delta\}\). Let \(f \in C(X)\) with \(Z(f^*) \supseteq \Delta\). Then \(f \in I\), whence \(fg \in I\) for all \(g \in C(X)\). But then

\[
\operatorname{cl}_{\beta X} Z(f) = \bigcap \{Z((fg)^*) : g \in C(X)\} \supseteq \Delta.
\]
(c) implies (d). Suppose that $S$ is far from $X$, but there exists $p \in \Delta$ with $p \notin \text{cl}_{\beta X} (\Delta \sim S)$. Then there exist $h, k \in C^*(X)$ such that $S \subseteq Z(h^*) \subseteq \beta X \sim X$, $p \notin Z(k^*)$ and $\Delta \sim S \subseteq Z(k^*)$. Let $f = hk$ and $g = 1/h$. Then $Z(f^*) = Z(h^*) \cup Z(k^*) \supseteq \Delta$, but $p \notin Z(k^*) = Z((fg)^*)$ so that $p \notin \text{cl}_{\beta X} Z(f)$.

(d) implies (a). Suppose that $\Delta$ is not an ideal set. Thus, if we let $I = \{ f \in C(X) : Z(f^*) \supseteq \Delta \}$, then there exist $f, g \in C(X)$ such that $f \in I$ and $fg \notin I$. So $Z(f^*) \supseteq \Delta$ and for some $p \in \Delta$, $p \notin Z((fg)^*)$. Let $Z$ be a zero-set neighborhood of $p$ in $\beta X$ such that $Z \cap Z((fg)^*) = \emptyset$; then $S = \Delta \cap Z$ is far from $X$, since $S \subseteq Z(f^*) \cap Z \subseteq \beta X \sim X$. But $Z$ is a neighborhood of $p$ which does not meet $\Delta \sim S$, so $\text{cl}_{\beta X} (\Delta \sim S) \neq \Delta$. □

It follows from 3.1 and 2.4 that every closed ideal is a $z$-ideal and therefore is absolutely convex (i.e. is an $l$-ideal; cf. [P, 3.7]).

It is clear from 3.1 that an ideal set must be close to $X$. The converse, however, does not hold. For example, let $X = \mathbb{N}$ and $\Delta = \{1, p\}$ where $p \in \beta N \sim N$. Then $\Delta$ is close to $X$, but is not an ideal set. We do have the following partial converse.

3.2 Lemma. Any compact subset of $\beta X$ which is close to $X$ contains an ideal set.

**Proof.** Let $K$ be a compact subset of $\beta X$ which is close to $X$, and define $\Delta = \cap \{ \text{cl}_{\beta X} Z(h) : Z(h^*) \supseteq K \}$, a nonvoid compact subset of $K$. We shall use 3.1(c) to show that $\Delta$ is an ideal set. Thus, suppose that $Z(f^*) \supseteq \Delta$ for some $f \in C(X)$. By the definition of $\Delta$, for each $p \in \beta X \sim Z(f^*)$, there exists $h \in C(X)$ such that $Z(h^*) \supseteq K$ and $p \notin \text{cl}_{\beta X} Z(h)$. Since $\beta X \sim Z(f^*)$ is an $F_\sigma$ in $\beta X$, it is Lindelöf, and therefore we can find $g_1, g_2, g_3, \ldots \in C(X)$ such that $Z(g_n^*) \supseteq K$ for each $n$, and $\cap_n \text{cl}_{\beta X} Z(g_n^*) \subseteq Z(f^*)$. Defining $g = \sum_{n=1}^\infty (1/2^n)(g_n^*)|_{A}$, we have $Z(g) = \cap_n Z(g_n) \subseteq Z(f)$ and $Z(g^*) = \cap_n Z(g_n^*) \supseteq K$. It follows that $\text{cl}_{\beta X} Z(f) \supseteq \text{cl}_{\beta X} Z(g) \supseteq \Delta$. □

4. Some corollaries. We now consider some consequences of 3.1 and 3.2. The first result follows also from [P, 2.6], where a more algebraic proof is given.

4.1 Corollary. If the ideal $I$ in $C(X)$ is contained in a unique maximal ideal (e.g., if $I$ is prime), then $I$ is closed if and only if $I$ is a real ideal.

**Proof.** Clearly a real ideal is closed [GJ, 8.4]. Suppose that $I$ is a closed ideal which is contained in the unique maximal ideal $M^p$ for some $p \in \beta X$. Then $O^* \subseteq I$ [GJ, 7.13], from which it follows that $\cap Z[I^*] = \{p\}$. Thus, $\{p\}$ is an ideal set, and it follows from 3.1 that, if $q \in \beta X \sim \nu X$, then $\text{cl}_{\beta X} (\{p\} \sim \{q\}) = \{p\} = \text{cl}_{\beta X} Z(p)$—i.e., $q \neq p$. Hence $p \in \nu X$ and $I = M^p$, a real ideal. □

It is clear that, for any nonvoid $E \subseteq X$, the set $\text{cl}_{\beta X} E$ is an ideal set.
4.2 Corollary. The following conditions are mutually equivalent.
(a) $X$ is Lindelöf.
(b) Every ideal set in $\beta X$ is of the form $\text{cl}_{\beta X} E$ for some $E \subseteq X$.
(c) Every ideal set in $\beta X$ meets $X$.
(d) Every closed ideal in $C(X)$ is an intersection of fixed maximal ideals.
(e) Every closed ideal in $C(X)$ is fixed.

Proof. (a) implies (b). Assume (a), let $\Delta$ be an ideal set in $\beta X$, and suppose that $p \in \beta X - \text{cl}_{\beta X}(\Delta \cap X)$. Let $F$ be a closed neighborhood of $p$ such that $F \cap (\Delta \cap X) = \emptyset$. Then $S = F \cap \Delta$ is a compact subset of $\beta X - X$, and hence by 3.1(d), $p \notin \text{cl}_{\beta X}(\Delta - S) = \Delta$. Hence $\Delta = \text{cl}_{\beta X}(\Delta \cap X)$, and (b) holds.

(b) implies (c). Obvious.

(c) implies (a). Assume that (a) is false, so there exists a compact subset $K$ of $\beta X - X$ which is close to $X$. By 3.2, $K$ contains an ideal set $\Delta$, so that (c) is false.

(b) if and only if (d). This follows easily from the fact that, for any $E \subseteq X$, $f \in \bigcap \{M_p : p \in E\}$ if and only if $\text{cl}_{\beta X} E \subseteq Z(f^*)$.

(c) if and only if (e). A closed ideal $\{f \in C(X) : Z(f^*) \supseteq \Delta\}$ is fixed if and only if $\Delta$ meets $X$. □

In 4.1 we showed that an ideal in $C(X)$ is a closed maximal ideal if and only if it is real. Clearly every closed maximal ideal is a maximal closed ideal. We conclude by proving the converse.

4.3 Corollary. An ideal in $C(X)$ is a closed maximal ideal if and only if it is a maximal closed ideal.

Proof. It suffices to show that every maximal closed ideal is a maximal ideal. Thus, suppose that $I$ is a maximal closed ideal in $C(X)$, and let $\Delta = \bigcap Z[I^*]$. If $\Delta$ is not a singleton set, say $q_1, q_2 \in \Delta$ with $q_1 \neq q_2$, then there exist compact sets $K_1$ and $K_2$ such that $K_1 \cup K_2 = \Delta$, $q_1 \notin K_1$ and $q_2 \notin K_2$. At least one of $K_1$ and $K_2$, say $K_1$, must be close to $X$. By 3.2, $K_1$ contains an ideal set $\Delta_1$. But since $q_1 \in \Delta - \Delta_1$, the closed ideal $J = \{f \in C(X) : Z(f^*) \supseteq \Delta_1\}$ is strictly bigger than $I$, contradicting the maximality of $I$. Therefore, we must have $\Delta = \{p\}$ for some $p \in \beta X$. But then $p \in vX$, and $I = M_p$, a maximal ideal. □

References


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