PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

FILTER CHARACTERIZATIONS OF C- AND C^*-EMBEDDINGS

JOHN WILLIAM GREEN

ABSTRACT. A filter F on a space S is completely regular if the complement of each set in F is completely separated from some set in F. A characterization of the Stone-Čech compactification due to Alexandroff is used to establish the following theorem. Suppose K is a subspace of a Tychonoff space S. K is C^*-embedded in S if and only if the trace on K of every maximal completely regular filter on S intersecting K is maximal completely regular on K. A similar characterization of the C-embedded subsets of a Tychonoff space is obtained as are several related results.

A characterization of the Stone-Čech compactification βS of a Tychonoff space S due essentially to Alexandroff [**1**] is used to characterize the C^*-embedded subspaces of S. This result is used to obtain a second characterization of such subspaces as well as one of the C-embedded subspaces. A few related results are obtained.

Throughout this paper, K will refer to a subspace of a Tychonoff space S. The notion of a completely regular filter was introduced in [**1**] under the term "completely regular system" and referred to a certain type of what is now called a filtersubbase. The term used here, as well as the reduction to filters, apparently was introduced by Bourbaki. (See, for example, [**4**, Chapter IX, §1, exercises].) The characterization of βS given below may be found, at least implicitly, in [**1**], [**3**], [**4**], [**5**], [**7**] and, particularly, [**9**]. In [**8**], as in several other papers, completely regular filters are used for distinct, though related, purposes. The reader is assumed to be familiar with the results in [**4**], as well as Chapter 6 of [**6**]. The terminology is that of these two sources, for the most part.

A filter F on S is completely regular if for each U in F, there exist V in F and ϕ in $L(S)$ (=the set of all functions in $C(S)$ with range a subset of $[0, 1]$) such that ϕ is 0 on V and 1 on $S-U$. It should be noted that every completely regular filter has as base an e-filter [**6**, problem 2L] and the filter (in the lattice of all subsets of S) generated by an e-filter is completely regular. If Y is the topology of S and for each $U \subseteq S$, $U^* = U \cup \{F{:}F$ is a

Presented to the Society, January 18, 1972; received by the editors October 21, 1971.
AMS 1970 *subject classifications*. Primary 54C45, 54D35.
Key words and phrases. Stone-Čech compactification, z-filter, completely regular filter, C^*-embedding, C-embedding.

free maximal completely regular filter on S having U as an element}, then $B=\{U^*: U \in Y\}$ is a base for a topology on S^* with respect to which S^* is (homeomorphic to) βS. If x is a point of a space T, $\mathrm{Nbd}_T(x)$ is the neighborhood filter of x in the space T. If F is a filter on S, F is said to intersect K if each set in F intersects K, and F_K and $\mathrm{Tr}_K(F)$ are used for the trace of F on K. A filterbase G is coarser than a filterbase F (written $G \leqq F$ or $F \geqq G$) if each set in G contains a set in F. If F and G are filters on a set T, $\sup\{F, G\}=\{U \subseteq T: U \supseteq f \cap g$ for some f in F and g in $G\}$ and is a filter on T, provided each set in F intersects each set in G.

LEMMA. *If F is a maximal completely regular filter on K, there is a unique maximal completely regular filter on S coarser than F. Furthermore, if F is any free completely regular filter on K, there is a coarser completely regular filter on S whose trace on K is free.*

PROOF. Suppose F is a free completely regular filter on K (relative to the subspace topology). Let $G=\{S-\mathrm{Cl}_s f: f \in F\}$. G is an S-open cover of K no finite subcollection of which covers K. For each x in K, let U_x denote some open set in G containing x and $\Phi_x=\{\phi \in L(S): \phi(x)=1$ and $\phi(S-U_x)=0\}$. For each finite collection H of ordered pairs (x, ϕ) such that $x \in K$ and $\phi \in \Phi_x$, let $\phi_H(t)=\sup\{\phi(t):(x, \phi) \in H\}$, for each t in S. $\phi_H \in L(S)$ and if $0<e<1$, then (1) $\phi_H^{-1}[0, e) \nsupseteq K$, for if $(x, \phi) \in H$, then $\phi(x)=1$; and (2) $\phi_H^{-1}[0, e) \cap K \neq \varnothing$, for otherwise, $K \subseteq \phi_H^{-1}[e, 1] \subseteq \phi_H^{-1}(0, 1] \subseteq \bigcup\{\phi^{-1}(0, 1]:(x, \phi) \in H\} \subseteq \bigcup\{U_x:(x, \phi) \in H\}$, contrary to the fact that no finite subcollection of G covers K. It follows that the filter F' on S with base $\{\phi_H^{-1}[0, e): 0<e<1$, H is a finite collection of ordered pairs (x, ϕ) such that $x \in K$ and $\phi \in \Phi_x\}$ is completely regular on S. It will be shown that $F' \leqq F$. Suppose $f' \in F'$. For some

$$H=\{(x_n, \phi_n): n \leqq p, x_n \in K, \phi_n \in \Phi_{x_n}\}$$

and $0<e<1$, $f' \supseteq \phi_H^{-1}[0, e)$. For each n, ϕ_n is 1 at x_n and for some $f_n \in F$, is 0 on $S-(S-\mathrm{Cl}_s f_n)=\mathrm{Cl}_s f_n \supseteq f_n$. $\bigcap_{n \leqq p} f_n=f \in F$. Thus, $\phi_n(f)=0$ for each $n \leqq p$, so $\phi_H(f)=0$. $f' \supseteq \phi_H^{-1}[0, e) \supseteq f$. Therefore, $F' \leqq F$.

Therefore, every free completely regular filter on K is finer than some (not necessarily free) completely regular filter on S whose trace on K is free. A simple application of Zorn's lemma establishes the existence of a filter F' maximal with respect to the property of being a completely regular filter on S coarser than F. F' is a maximal completely regular filter on S if F is on K. For suppose there is a completely regular filter G on S strictly finer than F'. $G \nleqq F$. $\sup\{G_k, F\}$ does not exist (as a filter), for if it does, it is a completely regular filter on K strictly finer than the maximal completely regular filter F on K. It follows that there exist g in G and f in F such that $\mathrm{Cl}_s g \cap \mathrm{Cl}_s f=\varnothing$. There exist g_1 in G and ϕ in $L(S)$

such that $\phi(g_1)=1$ and $\phi(S-g)=0$. Let $F''=\sup\{F', \{\phi^{-1}[0, e):0<e<1\}\}$. F'' is a completely regular filter on S strictly finer than F' and coarser than F. This is contrary to the definition of F'. Therefore, F' is a maximal completely regular filter on S. It is easily established that F' is unique. If F is a fixed maximal completely regular filter on K, then for some point x of K, $F=\mathrm{Nbd}_K(x)=\mathrm{Tr}_K(\mathrm{Nbd}_S(x))$.

THEOREM 1. *In order that K be C^*-embedded in S, it is necessary and sufficient that the trace on K of every maximal completely regular filter on S intersecting K be maximal completely regular on K.*

PROOF. The condition is sufficient. For suppose F is maximal completely regular on K and F' is the unique maximal completely regular filter on S coarser than F. It is easily seen that $F=F'_K$. Let $K'=K\cup \{F:F$ is a maximal completely regular filter on S and F_K is free$\}$. If $F\in K'-K$ and is fixed, F is the neighborhood system in S of some point of $S-K$ with which it will be identified. If $F\in K'-K$ and is free, then F is a point in $\beta S-S$ and $\{f^*:f\in F\}$ is a base for the neighborhood filter in βS of the point F. It is easily established that $K'=\mathrm{Cl}_{\beta S} K$ and hence is compact. Let $\phi:K'\to\beta K$ such that $\phi(x)=x$ if $x\in K$, $\phi(x)=\mathrm{Tr}_K(\mathrm{Nbd}_S(x))$ if $x\in K'\cap(S-K)$ and $\phi(x)=x_K$ if $x\in K'-S$. It is established above that ϕ is a bijection.

Suppose $x\in K'$ and U is a βK-open set containing $\phi(x)$.

Case 1. Suppose $x\in K$. There exists an S-open set D containing x such that $D^*(K)=D\cap K\cup\{F:F$ is a free maximal completely regular filter on K having $D\cap K$ as an element$\}\subseteq U$. $\phi(D^*\cap K')\subseteq D^*(K)$. For suppose $t\in D^*\cap S\cap K'$. $\phi(t)=t\in D^*(K)$. Suppose $t\in(D-K)\cap K'$ and $F=\mathrm{Nbd}_S(t)$. $\phi(t)=F_K$ and since $F\in D^*$, $F_K\in D^*(K)$. Suppose $t\in D^*\cap (K'-S)$. $\phi(t)=t_K$ and since $D\in t$, $D\cap K\in t_K$. Thus, $\phi(D^*\cap K')\subseteq D^*(K)\subseteq U$ and $x\in D^*\cap K'$.

Case 2. Suppose $x\in(K'-K)\cap S$. Let $F=\mathrm{Nbd}_S(x)$. $\phi(x)=F_K$. There exists f in F such that $f^*(K)\subseteq U$. $F_K\in f^*(K)$ and $F\in f^*$. That $\phi(f^*\cap K')\subseteq f^*(K)\subseteq U$ is established much as in Case 1.

Case 3. Suppose $x\in K'-S$. $\phi(x)=x_K$. There exists $f\in x$ such that $f^*(K)\subseteq U$. As in Case 2, $\phi(f^*\cap K')\subseteq f^*(K)\subseteq U$. Therefore, ϕ is continuous. A direct proof that ϕ^{-1} is continuous is not as simple, but homeomorphism is already established without that. So, $\beta K\subseteq\beta S$ and K is C^*-embedded in S.

The condition is necessary. For in this case, $\beta K\subseteq\beta S$. If F is a maximal completely regular filter on S fixed at a point x of K, then $F=\mathrm{Nbd}_S(x)$ and $F_K=\mathrm{Nbd}_K(x)$, which is maximal completely regular on K. Suppose F is a maximal completely regular filter on S intersecting K such that F_K is free. There is a maximal completely regular filter on K finer than the completely regular filter F_K. Suppose there are two, G_1 and G_2. G_1 and G_2 converge to distinct points of βK. Hence, F accumulates at two points of βS, which is impossible. Let F' denote the unique maximal completely regular filter on K finer than F_K. Suppose $F' \neq F_K$. Then there is a set f' in F', open in K, and containing no set in F_K. Thus, for every closed g in F, $g \cap K - f'$ is a nonempty set closed in K. Since βK is compact, $\bigcap \{\mathrm{Cl}_{\beta K}\, g \cap K - f' : g = \mathrm{Cl}_S\, g \in F\}$ contains a point, P, which is a βS-accumulation point of F but not of F'. F' converges in βK to $F' \neq P$. It follows that F accumulates at the two points P and F', which is impossible. Therefore, $F_K = F'$.

COROLLARY. *If K is a discrete subspace of S, then K is C^*-embedded in S if and only if the trace on K of every maximal completely regular filter on S intersecting K is an ultrafilter on K.*

THEOREM 2. *In order that K be C^*-embedded in S, it is necessary and sufficient that every maximal completely regular filter on K be the trace on K of a maximal completely regular filter on S.*

PROOF. The condition is sufficient. For suppose F is a maximal completely regular filter on S intersecting K. F_K is completely regular on K, so there exists a maximal completely regular filter G on S such that G_K is finer than F_K and is maximal completely regular. Since G_K and F_K are compatible, so are F and G; and since F and G are maximal, $F = G$. Thus, F_K is maximal completely regular and the stated result follows from Theorem 1.

The necessity of the condition follows easily from Theorem 1 and the lemma.

THEOREM 3. *If K is C^*-embedded in S, the trace on K of every z-ultrafilter on S intersecting K is a z-ultrafilter on K.*

PROOF. Suppose J is a z-ultrafilter on S intersecting K. Let F denote the unique maximal completely regular filter on S coarser than J. $F_K \leqq J_K$ and by Theorem 1 is maximal completely regular on K. There is a unique z-ultrafilter Q on K finer than F_K. Suppose there exist U in J_K and V in Q such that $U \cap V = \varnothing$. Then there exists $\phi \in L(K)$ such that $\phi^{-1}(0) = U$ and $\phi^{-1}(1) = V$. ϕ has a continuous extension ϕ_1 in $L(S)$. $\phi_1^{-1}[0, 1) \in F$ since each set in F intersects $\phi_1^{-1}[0, \frac{1}{2})$. Thus, the subset U of $\phi_1^{-1}(1)$ fails to intersect some set in F_K and yet $J_K \geqq F_K$. This is a contradiction. Thus, each set in Q intersects each set in J_K. Since Q is a z-ultrafilter on K, $Q \supseteq J_K$. Suppose $V \in Q$. There exists ϕ in $L(K)$ such that $\phi^{-1}(0) = V$. ϕ has a continuous extension ϕ_1 in $L(S)$. Since V intersects every set in J_K, $\phi_1^{-1}(0)$ intersects every set in J and thus belongs to J. Hence, $\phi_1^{-1}(0) \cap K = V \in J_K$. It follows that $Q = J_K$.

The converse of the above theorem is false, even if the closure in S of every zero set in K is a zero set in S. In this regard, Lemma 3 of **[2]** may be of interest, where the normal base is the collection of all zero sets.

EXAMPLE. Let $S=[0, 1]$, $K=[0, 1)$ with the usual topologies. Obviously, K is not C^*-embedded in S. The only z-ultrafilters on S intersecting K are those fixed at a point of K. If Z is a zero set in K, then $\mathrm{Cl}_S Z$ is a zero set in S since it is closed and S is metric.

THEOREM 4. *In order that K be C-embedded in S, it is necessary and sufficient that every z-ultrafilter on K be the trace of a z-ultrafilter on S.*

PROOF. The condition is necessary. For by Theorems 1 and 3, $\beta K \subseteq \beta S$, and the trace on K of every z-ultrafilter on S intersecting K is a z-ultrafilter on K. Suppose F is a z-ultrafilter on K. Let G denote the unique maximal completely regular filter on S coarser than F, so that G_K is the unique maximal completely regular filter on K coarser than F. Let J denote the unique z-ultrafilter on S finer than G. Suppose some set U in J does not intersect K. Since K is C-embedded in S, there exists g in $L(S)$ such that $g^{-1}(0) \supseteq K$ and $g^{-1}(1) \supseteq U$. For each e in $(0, 1)$, $g^{-1}[0, e) \in G$ and hence, $J \ngeqq G$. This is a contradiction. Thus, J intersects K and J_K is a z-ultrafilter on K. Since $J \geqq G$, $J_K \geqq G_K$. There is only one z-ultrafilter on K finer than G_K. Hence, $J_K = F$.

The condition is sufficient. It will first be shown that K is C^*-embedded in S. It follows easily from the hypothesis that the trace on K of every z-ultrafilter on S intersecting K is a z-ultrafilter on K. Suppose F is a maximal completely regular filter on K. Let J denote the unique z-ultrafilter on S such that $J_K \geqq F$. Let G denote the unique maximal completely regular filter on S coarser than J. There exists a maximal completely regular filter T on K finer than G_K and a unique z-ultrafilter Q on S such that $Q_K \geqq T$. From the first lemma, there is only one maximal completely regular filter on S coarser than T and $T \geqq G_K \geqq G$. Thus, G is that unique filter. Since $Q_K \geqq T$, $Q \geqq G$. Thus, Q and J are z-ultrafilters on S finer than G. It follows that $Q=J$ and $Q_K = J_K$ and $T=F$. It follows that F is the only maximal completely regular filter on K finer than G_K. Suppose $F \neq G_K$. Then there exists f in F such that for every g in G_K, $g-f \neq \varnothing$. There exist f_1 in F and $\phi \in L(K)$ such that $\phi(f_1)=1$ and $\phi(K-f)=0$. Thus, if $0<e<1$, $\phi^{-1}[0, e) \cap f_1 = \varnothing$, but if $g \in G_K$, $\phi^{-1}[0, e) \cap g \neq \varnothing$. There is a z-ultrafilter W on S such that

$$W_K \geqq H = \sup\{G_K, \{\phi^{-1}[0, e) : 0 < e < 1\}\}.$$

Since H and F are incompatible, W_K and F are also. But since $W_K \geqq H \geqq G_K$, it follows that $W \geqq G$ and $W=J$. This is contrary to the incompatibility of W_K and $J_K \geqq F$. Therefore, $F = G_K$. By Theorem 2, K is C^*-embedded in S.

Suppose K is not C-embedded in S. From Theorem 1.18 of [6], it follows that there is a zero set Z in S not intersecting K such that if $g \in C^*(S)$ and $g^{-1}(0)=Z$, then for each $e>0$, $g^{-1}[0, e]\cap K\neq\varnothing$. Let $F_1=\{g^{-1}[0, e)\cap K: 0<e, g \in C^*(S)$ and $g^{-1}(0)=Z\}$. F_1 is a base for a z-filter on K. Hence, there is a z-ultrafilter F on K finer than F_1. F is the trace on K of some z-ultrafilter J on S, by hypothesis. $Z \notin J$, since $Z\cap K=\varnothing$, so there exists $V \in J$ such that $V\cap Z=\varnothing$. Since Z and V are zero sets in S, there exists g in $L(S)$ such that $g^{-1}(0)=Z$ and $g^{-1}(1)=V$. But if $0<e<1$, $g^{-1}[0, e]\cap K \in F\subseteq J_K$ and thus, $g^{-1}[0, e]\cap V\neq\varnothing$. This is a contradiction. Therefore, K is C-embedded in S.

A minor modification of the argument in the last paragraph above establishes the following.

THEOREM 5. *If K is C^*-embedded in S and every z-ultrafilter on K is finer than some z-ultrafilter on S, then K is C-embedded in S.*

The following summary of Theorems 2 and 4 was suggested by the referee. It should be noted, however, that while the trace of a completely regular filter on S on an arbitrary subset K is completely regular on K, the same is not true of e-filters without some restriction on K.

THEOREM 6. *K is C- [C^*-] embedded in S if and only if every z- [e-] ultrafilter on K is the trace of a z- [e-] ultrafilter on S.*

THEOREM 7. *If K is countable, then K is C-embedded in S if and only if K is completely separated from every zero set in S not intersecting K.*

PROOF. Suppose K is completely separated from every zero set in S not intersecting K. It follows from 3B.1 of [6] that K is closed and completely separated from every closed set not intersecting S. Suppose K_1' and K_2' are subsets of K completely separated in K. There exists ϕ in $L(K)$ such that $\phi(K_1')=0$ and $\phi(K_2')=1$. Since K is countable, there exists $0<r<1$ such that $\phi^{-1}(r)\cap K=\varnothing$. It follows that $K_1=K\cap\phi^{-1}[0, r]$ and $K_2=K\cap\phi^{-1}[r, 1]$ are completely separated in K, contain K_1' and K_2' respectively, and $K=K_1\cup K_2$.

Every closed subset of K is the intersection of K and a zero in S. For suppose H is a closed subset of K. For each x in $K-H$, there is a zero set Z_x in S containing H but not containing x. $\bigcap Z_x$ is the intersection of countably many zero sets in S and thus is a zero set whose intersection with K is H.

Thus, there exist zero sets Z_1 and Z_2 in S such that $Z_1\cap K=K_1$ and $Z_2\cap K=K_2$. $Z_1\cap Z_2$ is a zero set not intersecting K and so, by hypothesis, there is a zero set Z in S containing K and not intersecting $Z_1\cap Z_2$. $Z\cap Z_1$ and $Z\cap Z_2$ are mutually exclusive zero sets in S containing K_1' and

K_2', respectively. Hence, each two sets completely separated in K are completely separated in S. By Urysohn's extension theorem, K is C^*-embedded in S. It follows from Theorem 1.18 of [6] that K is C-embedded in S. That the converse is true is obvious.

Thus, statements 1 and 3 of problem 3L.4 of [6] remain equivalent even if the requirement that D be discrete is omitted.

REFERENCES

1. P. S. Aleksandrov, *Bikompakte Erweiterung topologischer Räume*, Mat. Sb. **5** **(47)** (1939), 403–423. (Russian) MR **1**, 318.

2. R. A. Alo and H. L. Shapiro, *$\mathscr{L}$-realcompactifications and normal bases*, J. Austral. Math. Soc. **9** (1969), 489–495. MR **39** #3455.

3. B. Banaschewski, *Über zwei Extremaleigenschaften topologischer Räume*, Math. Nachr. **13** (1955), 141–150. MR **17**, 66.

4. N. Bourbaki, *Éléments de mathématique*. Part. 1. *Les structures fondamentales de l'analyse*. Livre III: *Topologie générale*, Actualités Sci. Indust., no. 1029, Hermann, Paris, 1947; English transl., Addison-Wesley, Reading, Mass., 1966. MR **9**, 261; MR **34** #5044b.

5. Ky Fan and N. Gottesman, *On compactifications of Freudenthal and Wallman*, Nederl. Akad. Wetensch. Proc. Ser. A **55**=Indag. Math. **14** (1952), 504–510. MR **14**, 669.

6. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960. MR **22** #6994.

7. O. Njästad, *A note on compactification by bounding systems*, J. London Math. Soc. **40** (1965), 526–532. MR **33** #1832.

8. R. M. Stephenson, Jr., *Product spaces for which the Stone-Weierstrass theorem holds*, Proc. Amer. Math. Soc. **21** (1969), 284–288. MR **40** #3499.

9. F. J. Wagner, *Notes on compactification*. I, II, Nederl. Akad. Wetensch. Proc. Ser. A **60**=Indag. Math. **19** (1957), 171–176, 177–181. MR **19**, 436.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73069