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TOROIDAL ARCS ARE CELLULAR[1]

TOM KNOBLAUCH

ABSTRACT. We prove that a toroidal, cell-like, locally connected continuum is cellular.

1. **Introduction.** An arc may be a decreasing intersection of cubes-with-two-handles and still not be cellular, or even toroidal. An arc formed by joining two Fox-Artin arcs [4] at their tame ends serves as an example.

However it follows directly from the theorem below that all toroidal arcs are cellular. The theorem generalizes Daverman's result [3] concerning toroidal 3-cells. It was suggested to me as a problem by D. R. McMillan.

2. **Definitions.** A continuum X in a 3-manifold M^3 is *cellular* if $X=\bigcap_{i=1}^{\infty} X_i$ where $X_{i+1}\subseteq \operatorname{Int} X_i$ for each i, and each X_i is a 3-cell in M^3.

A continuum X in M^3 is *toroidal* if $X=\bigcap_{i=1}^{\infty} X_i$ where $X_{i+1}\subseteq \operatorname{Int} X_i$ for each i, and each X_i is a solid torus in M^3.

A continuum X in M^3 is *cell-like* if for any neighborhood U of X in M^3, there is a neighborhood V of X in M^3 such that V is homotopically trivial in U.

3. THEOREM. *A toroidal, cell-like, locally connected continuum is cellular.*

PROOF. Assume the continuum X is toroidal and cell-like but not cellular. Then $X=\bigcap_{i=0}^{\infty} T_i$ where for each i, T_i is a solid torus and $T_{i+1}\subseteq \operatorname{Int} T_i$. Since X is cell-like, we may assume that for each i the winding number of T_{i+1} in T_i is zero (that is, T_{i+1} is homotopically trivial in T_i). Since X is not cellular, we may assume that for each i the wrapping number of T_{i+1} in T_i is not zero (that is, each meridional disk of T_i intersects T_{i+1}). Let D, E, F, and G be four disjoint polyhedral meridional disks of T_0 with F and G in different components of $T_0-(D\cup E)$. We may assume that for each i, T_i is polyhedral and Bd T_i is in general position with $\Delta=D\cup E\cup F\cup G$. $\Delta\cap \operatorname{Bd} T_i$ is a finite collection of trivial and meridional (with respect to Bd T_i) simple closed curves, because the wrapping number of T_i in T_0 is not zero [**2**, Theorem 1].

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If $\Delta \cap \text{Bd } T_1$ has any trivial simple closed curves on Bd T_1, choose J to be an innermost trivial curve on Bd T_1. J bounds a disk D' on Bd T_1 whose interior misses Δ.

J lies in one of the four disks D, E, F, or G, say D, and bounds a disk D'' there. Replace D by $(D-D'')\cup D'$ and then push D' slightly off Bd T_1 to the appropriate side. We can remove all trivial curves in this way. We have four new disjoint meridional disks D_1, E_1, F_1, and G_1.

Similarly change (D_1, E_1, F_1, G_1) to (D_2, E_2, F_2, G_2) so that if $\Delta_2 = D_2\cup E_2\cup F_2\cup G_2$ then $\Delta_2\cap \text{Bd } T_j$ contains no trivial simple closed curves of Bd T_j for $j\leqq 2$.

Continue this process to get a sequence $(D_1, E_1, F_1, G_1), \cdots, (D_n, E_n, F_n, G_n), \cdots$ where for each n, if $\Delta_n = D_n\cup E_n\cup F_n\cup G_n$ then $(\text{Bd } T_j)\cap\Delta_n$ contains no trivial curves of Bd T_j for $J\leqq n$.

We use this construction to prove the following lemma.

LEMMA. *There are infinitely many components of* $X-\Delta$ *each of whose closures intersect two of the disks* D, E, F, *and* G.

PROOF. It is clearly enough to show that given $n>0$ there are at least n such components. It is also enough to show that given $n>0$, $\exists m>0$ such that there are at least n components of $X-\Delta_m$ whose closures intersect two of D_m, E_m, F_m, and G_m. In fact, each component of $X-\Delta_m$ whose closure intersects, say, D_m and F_m contains a component of $X-\Delta$ whose closure intersects D and F. To see this let C be a component of $X-\Delta_m$ such that $\bar{C}$ intersects D_m and F_m. Then $\bar{C}$ intersects D and F since $\Delta_m\cap X\subseteq \Delta\cap X$. By a theorem of elementary topology $\bar{C}$ contains an irreducible continuum C' from D to F and $C'-(D\cup F)$ is connected. The component of $X-\Delta$ containing $C'-(D\cup F)$ lies in C and its closure intersects D and F.

Now, fixing $N>0$, consider a homotopy core J of T_N lying in Bd T_N. Also take J so that it intersects each curve of $(\text{Bd } T_N)\cap\Delta_N$ just once. J must contain a subarc from D_N to E_N. Without loss of generality assume this arc lies in the F_N half of $T_0-(D_N\cup E_N)$. The existence of this arc assures the existence of a cylinder in Bd T_N with one end in D_N and one in E_N. A *spanning cylinder* is an annulus with interior in the F_N half of $T_0-(D_N\cup E_N)$ and with one boundary component in D_N and one in E_N. Spanning cylinders are defined only for the integer N. A spanning cylinder A is said to be *inside* a spanning cylinder B if $D_N\cup E_N\cup B$ separates Int A from Bd T_0. *Inside* B is the bounded closed component of $T_0-(D_N\cup E_N\cup B)$. Choose an outermost spanning cylinder $C_{N,1}\subseteq \text{Bd } T_N$. $C_{N,1}$ must lie inside an outermost spanning cylinder $C_{N-1,1}\subseteq \text{Bd } T_{N-1}$. The following linking argument assures the existence of another outermost spanning cylinder $C_{N,2}\subseteq \text{Bd } T_N$ inside $C_{N-1,1}$.

Suppose $C_{N,1}$ is the unique outermost spanning cylinder of Bd T_N inside $C_{N-1,1}$. Let $J_1 = C_{N,1} \cap D_N$ and $J_2 = C_{N-1,1} \cap D_N$. Pull J slightly off $C_{N,1}$ to the inside of T_N. Then we still have $J \subseteq T_N$. J_1 is a meridian of Bd T_N, so the disk in D_N bounded by J_1 contains an odd number of points of J. J_2 is a meridian of Bd T_{N-1}, so since the winding number of T_N in T_{N-1} is zero, the disk in D_N bounded by J_2 contains an even number of points of J. Then the annulus in D_N bounded by J_1 and J_2 contains an odd number of points of J. However, by the uniqueness assumption, each point of J in the annulus is the endpoint of subarc of J which runs from D_N into the F_N half of $T_0 - (D_N \cup E_N)$ and back to D_N again. Therefore the number of points of J in the annulus is even, a contradiction.

Now $C_{N-1,1}$ is inside an outermost spanning cylinder $C_{N-2,1}$ of T_{N-2}. The linking argument gives us another outermost spanning cylinder $C_{N-1,2}$ of Bd T_{N-1} inside $C_{N-2,1}$. After N applications of the linking argument we have spanning cylinders $C_{i,j}$, $1 \leqq i \leqq N$ and $j \leqq 2$, where each $C_{i,j} \subseteq$ Bd T_i and is an outermost such cylinder. In addition $C_{i,j}$ is inside $C_{i-1,1}$ for $1 < i \leqq N$. Thus

$$\text{inside } C_{i,2} \cap \text{inside } C_{j,2} = \varnothing \quad \text{for } i \neq j.$$

The lemma will be proved if we can find a component of $X - \Delta_N$ inside each $C_{i,2}$ whose closure intersects F_N and one of D_N or E_N. If we knew $X \cap F_N \cap \text{inside } C_{i,2} \neq \varnothing$, then we could find an irreducible continuum C' in X from $F_N \cap \text{inside } C_{i,2}$ to $(D_N \cup E_N) \cap \text{inside } C_{i,2}$. The component of $X - \Delta_N$ containing $C' - \Delta_N$ would be the desired component. Then we need only show $X \cap F_N \cap \text{inside } C_{i,2} \neq \varnothing$. Take an innermost (in F_N) simple closed curve J of $F_N \cap C_{i,2}$. J is meridional on T_i and bounds a disk H in F_N which intersects X only if F_N does inside $C_{i,2}$. Choose an innermost (in H) curve J' of $H \cap \text{Bd } T_i$. J' bounds a disk H' in H and inside $C_{i,2}$. H' is a meridional disk of T_i so $H' \cap X \neq \varnothing$ and therefore $X \cap F_N \cap$ inside $C_{i,2} \neq \varnothing$ and the lemma is proved.

The theorem now follows easily from the lemma. There are infinitely many components of $X - \Delta$ whose closures intersect two of these disks, say D and F. It follows easily that X is not locally connected.

4. Corollary. *Suppose X is a cell-like, locally connected continuum in S^3 and $X = \bigcap_{i=0}^{\infty} X_i$ where $X_{i+1} \subseteq \text{Int } X_i$ and each X_i is a 3-manifold bounded by a torus (a solid torus or a cube-with-a-knotted-hole). Then X is cellular.*

Proof. Assume X satisfies the hypotheses of the corollary but X is not cellular. By the theorem X is not toroidal, so we may assume each X_i is a cube-with-a-knotted-hole [**1**]. Since X is cell-like we may assume X_{i+1} is homotopically trivial in X_i for each i. Since X is not cellular we may assume no X_i contains a cell with X in its interior. Also take each X_i to be polyhedral.

Now let $T_i = S^3 - \text{Int } X_i$ for each i. Then $T_i \subseteq \text{Int } T_{i+1}$ and each T_i is a knotted solid torus (this is the definition of a cube-with-a-knotted-hole).

Property 1. For each i, there is no meridional disk of T_{i+1} missing T_i. For if there were such a polyhedral disk D, a closed regular neighborhood $N(X_{i+1} \cup D)$ would be a cell in X_i containing X in its interior.

Property 2. T_i is homotopically trivial in T_{i+1}, or $T_i \sim 0$ in T_{i+1}. Since $X_{i+1} \sim 0$ in X_i, if J is a meridional simple closed curve of T_{i+1}, then $J \sim 0$ in $S^3 - T_i$. So the winding number of T_i in T_{i+1} is zero, so $T_i \sim 0$ in T_{i+1}.

However Kister and McMillan [5] showed that the union of an ascending sequence of knotted solid tori with the Properties 1 and 2 cannot be imbedded in S^3. The corollary is thus proved by contradiction.

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Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706