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KNOTS WITHOUT UNKNOTTED INCOMPRESSIBLE SPANNING SURFACES

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ABSTRACT. We construct a tame knot in S^3 which has no unknotted incompressible spanning surface.

I. **Introduction.** Schaufele [**6**] has exhibited tame knots in S^3 which have both knotted minimal spanning surfaces and unknotted incompressible spanning surfaces of higher genus. The Seifert construction [**1**, p. 140] yields an unknotted spanning surface for every knot. These observations led us to ask [**3**, p. 42] if every knot has an unknotted incompressible spanning surface. The answer is negative, with a counterexample provided by the knot in Figure 2, and the same techniques can be used to construct more counterexamples. Thus we establish the existence of a new class of knots; viz., those knots which have unknotted incompressible spanning surfaces. This new class lies properly between the Neuwirth knots and all knots. The Zt module structure of the abelianized commutator subgroup of our knot is isomorphic to that of the square knot. Hence it is impossible to obtain an algebraic characterization of this new class of knots in terms of many of the classical tools of knot theory; e.g., Alexander matrices. Rice [**5**] has done some preliminary work with this class of knots.

The knot in Figure 2 also provides a counterexample to a conjecture of [**3**, p. 44]. All work is done in the PL category, all surfaces are orientable and properly embedded, all knots are in S^3, and the terminology and notation follow that of [**3**] and [**4**].

II. **The construction.** Let Q be the knot space of the nontrivial torus knot (p, q), with longitude λ and unique [**2**] meridian μ.

LEMMA. *If* $B \subset Q$ *is a connected, incompressible, and boundary incompressible bounded surface, then no component of* Bd B *is isotopic, in* Bd Q, *to* μ.

PROOF. If we assume the contrary, then Bd B lies in an annulus $A \subset \text{Bd } Q$, and each component of Bd A is isotopic, in Bd Q, to μ. Let

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Q_1 and Q_2 be two copies of Q, with subscripts identifying respective subspaces. If we join Q_1 and Q_2 by identifying A_1 with A_2 and $B_1 \cap A_1$ with $B_2 \cap A_2$, then the closed incompressible surface $B_1 \cup B_2$ must separate the resulting knot space. Thus B separates Q, so $\pi_1(Q)$ can be written as a nontrivial free product with amalgamation along $\pi_1(B)$. But $\pi_1(Q)$ has nontrivial center, so $\pi_1(B)=Z$; i.e., B is an annulus. This contradicts the fact that the torus knot (p, q) is prime. □

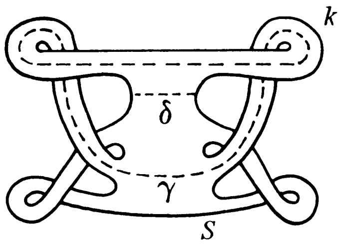

FIGURE 1

Consider the square knot k and its spanning surface S as shown in Figure 1. The manifold $\mathrm{Cl}(S^3-N(k))$ is fibered over S^1, with $S\cap \mathrm{Cl}(S^3-N(k))$ as fiber. Note that the curve γ is chosen so that $N(\gamma)\cap S$ is an unknotted and untwisted annulus. Thus there is a properly embedded nonsingular disk $D\subset \mathrm{Cl}(S^3-N(\gamma))$ with $(\mathrm{Bd}\, D)\cap S=\varnothing$. This disk D may also be chosen so that $D\cap S=\delta\subset \mathrm{Int}\, D$, where $\delta\subset S$ is the spanning arc indicated in Figure 1. The surface $\mathrm{Bd}\, N(\gamma)$ is incompressible in $\mathrm{Cl}(S^3-(N(k)\cup N(\gamma)))$. Let α be the oriented boundary of a nonseparating disk in $N(\gamma)$, and let β be the oriented curve $D\cap \mathrm{Bd}\, N(\gamma)$, so that α and β form a basis for $\pi_1(\mathrm{Bd}\, N(\gamma))=Z\times Z$.

Remove $\mathrm{Int}\, N(\gamma)$ from S^3 and replace it with Q sewn in so that α is identified with λ and β with μ. The disk D is sewn along μ, so the resulting manifold V is still homeomorphic to S^3, and $\mathrm{Cl}(V-N(k))=M$ is a knot space. The corresponding knot, for the case $(p, q)=(2, 3)$, is shown in Figure 2.

THEOREM 1. *If T is any incompressible spanning surface in the knot space M, then T is knotted.*

PROOF. We prove this by moving T with an isotopy until it misses Q. The surface S separates $\mathrm{Bd}\, N(\gamma)=\mathrm{Bd}\, Q$ into two annuli, whose closures we denote by A_1 and A_2. Each component of $A_1\cap A_2$ is a meridian in $\mathrm{Bd}\, Q$. If $S'=(S\cap M)\cup A_1$, then S' is also a spanning surface in M. Put T in general position with S' and assume the number of components in $S'\cap T$ is minimal.

If $S' \cap T \neq \varnothing$, the geometric structure of the covering corresponding to the commutator subgroup of $\pi_1(M)$ assures us that there is some component of $T \cap (M - S')$ whose closure T' meets S' only from the side opposite A_2. Put T' in general position with A_2 and assume the number of components in $T' \cap A_2$ is minimal. Now $M - (S' \cup Q)$ was constructed so as to have a product structure, so if $T' \cap A_2 = \varnothing$, we know from Waldhausen [7] that T' is parallel to a surface in S'. Hence we can move T by an isotopy which reduces the number of components in $T \cap S'$, a contradiction.

If $T' \cap A_2 \neq \varnothing$, then each component of $T' \cap A_2$ is a meridian in Bd Q. Put T in general position with Bd Q without moving $T' \cap A_2$. Our lemma assures us that each component of $T \cap Q$ is boundary parallel and hence may be removed by an isotopy. If $S' \cap T = \varnothing$, then $T \cap A_2$ may be assumed to be either empty or to consist entirely of meridians in Bd Q, and similar reasoning allows us to remove each component of $T \cap Q$. □

COROLLARY. *Let T_i, $1 \leqq i \leqq n$, be a set of pairwise disjoint incompressible spanning surfaces in M. Then some component of* $\mathrm{Cl}(M - \bigcup_{i=1}^{n} N(T_i))$ *is not a cube-with-handles.* □

Thus we have a counterexample to our conjecture [3, p. 44] that every knot space M contains a finite system T_i, $1 \leqq i \leqq n$, of pairwise disjoint incompressible spanning surfaces such that:

(a) Each component of $\mathrm{Cl}(M - \bigcup_{i=1}^{n} N(T_i))$ is a cube-with-handles, and

(b) any additional disjoint incompressible spanning surface is parallel to one of the T_i's.

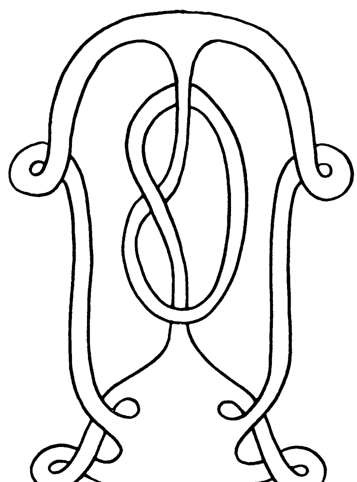

FIGURE 2

Let $G_1=\pi_1(M)$, let G_2 be the group of the square knot, let $G'_i=[G_i, G_i]$, and let $G''_i=[G'_i, G'_i]$ for $i=1, 2$.

THEOREM 2. *The Zt modules G'_i/G''_i are isomorphic, for $i=1, 2$.*

PROOF. We saw in the proof of Theorem 1 that an incompressible spanning surface for M may be assumed to miss Q. Hence, if $\varphi:\tilde{M}\to M$ denotes the covering of M corresponding to G'_1, then φ is a homeomorphism when restricted to any component of $\varphi^{-1}(Q)$. But M was constructed in such a way that if we remove Q and sew in a solid torus W' so that α bounds a nonseparating disk in W', we will have the square knot space. The corresponding operation lifted to $\tilde{M}$ converts $\tilde{M}$ into the covering of the square knot space corresponding to G'_2. In either case, the curves $\varphi^{-1}(\beta)$ will generate all of the homology contributed by either $\varphi^{-1}(Q)$ or $\varphi^{-1}(W')$, and the curves $\varphi^{-1}(\alpha)$ will all be homologous to zero. □

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