PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

SMOOTH STRUCTURE AND SIGNATURE OF CODIMENSION 2 EMBEDDINGS

DIETER ERLE

ABSTRACT. In an earlier paper (Topology **8** (1969), 99–114), the author defined a signature for codimension 2 embeddings in S^{4k+1} and proved this signature to be an invariant of the topological type of the embedding. For embedded homotopy $(4k-1)$-spheres, this signature is known to detect the smooth structure. It turns out that it determines the smooth structure also for integer homology $(4k-1)$-spheres and, by a result of A. Durfee, for closed $(2k-2)$-connected $(4k-1)$-manifolds with finite $(2k-1)$-dimensional homology. As a consequence, in the above cases, the smooth structure is given by the topological type of the embedding. On the other hand, for $k=3, 4, 5, 7, 15$, we exhibit examples of $(4k-1)$-manifolds embedded in a $(4k+1)$-sphere for which the smooth structure is not determined by the signature of the embedding.

In [7] we defined a signature σ_1 for smooth embeddings $M^{4k-1}\subset S^{4k+1}$ where M is an oriented closed $(4k-1)$-manifold with finite $(2k-2)$- and $(2k-1)$-dimensional integer homology. M bounds an oriented (and therefore parallelizable) manifold F in S^{4k+1}, and σ_1 equals $\sigma(F)$, the signature of F [7, Korollar 5.3]. σ_1 was proved to be an invariant of the topological type of the embedding [7, Satz 5.2]. Here, two embeddings $M_1\subset S^{4k+1}$ and $M_2\subset S^{4k+1}$ are said to be of the same topological type if there exists an orientation preserving homeomorphism $h:S^{4k+1}\to S^{4k+1}$ with $h(M_1)=M_2$ and $h|M_1:M_1\to M_2$ orientation preserving. As σ_1 detects the smooth structure of M if M is a homotopy sphere $(k\geqq 2)$, the smooth structure is an invariant of the topological type of the embedding for ordinary knots (i.e. embedded homotopy spheres) in these dimensions. The following general question arises: Given a manifold M other than a homotopy sphere, with some smooth structure embeddable with codimension 2, for which the signature σ_1 of any such embedding is defined and is an invariant of the topological type. Does the signature determine the smooth structure of M? Theorems 1 and 2 provide a partial answer to this question.

Received by the editors February 1, 1972.

AMS 1970 *subject classifications*. Primary 57D40, 57D55; Secondary 57C45.

Key words and phrases. Smooth structure, signature, manifold, higher dimensional knots, codimension 2, topological type, embedding, homology sphere, highly connected manifold.

THEOREM 1. *Let M^{4k-1} be an oriented closed $(4k-1)$-manifold smoothly embedded in S^{4k+1}. Assume that M is either*

(i) *an integer homology sphere of dimension $4k-1$, $k \geqq 2$, or*

(ii) *a $(2k-2)$-connected $(4k-1)$-manifold with finite integer homology in dimension $2k-1$, $k \geqq 3$, $k \neq 4$.*

Then the signature σ_1 of the embedding $M \subset S^{4k+1}$ determines the smooth structure of M.

COROLLARY. *Under the assumptions of Theorem 1, the smooth structure of M is an invariant of the topological type of the embedding.*

Using the properties of the signature σ_1 above, Theorem 1 clearly is a consequence of the following proposition.

PROPOSITION. *Let N_1 and N_2 be smooth parallelizable oriented $(4k)$-dimensional manifolds with boundary such that ∂N_1 and ∂N_2 are orientation preservingly homeomorphic to each other and are either*

(i) *integer homology spheres of dimension $4k-1$, $k \geqq 2$, or*

(ii) *$(2k-2)$-connected $(4k-1)$-manifolds with finite integer homology in dimension $2k-1$, $k \geqq 3$, $k \neq 4$.*

Then ∂N_1 and ∂N_2 are orientation preservingly diffeomorphic to each other if and only if $\sigma(N_1) \equiv \sigma(N_2) \bmod 8\omega_{4k}$ (where ω_{4k} is the order of bP_{4k}).

Part (ii) of this Proposition was proved by A. Durfee [**6**, Theorem 8.1 and Proposition 8.6] making use of sophisticated algebraic tools. We give an independent proof under the additional assumption that

$$(*) \qquad H_{2k-1}(\partial N_1; Z) \text{ has no 2-torsion if } k \equiv 0 \text{ or } 1 \bmod 4.$$

Our proof is topological, in fact it is a straightforward generalization of Kervaire's and Milnor's proof for the case of homotopy spheres [**8**], based on a paper by Wall [**17**].

PROOF OF THE PROPOSITION (ASSUMING (*)). We first have to establish that ∂N_1—point and ∂N_2—point are diffeomorphic. If ∂N_i is a homology sphere, ∂N_i—point is acyclic. Therefore it has only one combinatorial structure [**9**], and by [**13**] only one smooth structure, which takes care of case (i). Almost closed manifolds satisfying (ii) have been studied by Wall [**17**]. In his notation, all tangential invariants of ∂N_i—point are trivial because ∂N_i—point is parallelizable. As ∂N_1—point and ∂N_2—point also have the same homological invariants, they are orientation preservingly diffeomorphic by [**17**, p. 284]. Thus we know that ∂N_1 is orientation preservingly diffeomorphic to $\partial N_2 \# \Sigma$ for a suitable homotopy sphere Σ.

Now suppose ∂N_1 is orientation preserving diffeomorphic to ∂N_2. Construct a smooth manifold N from the disjoint union of N_1 and N_2 by matching ∂N_1 with ∂N_2 by some orientation preserving diffeomorphism.

$\sigma(N)=\sigma(N_1)-\sigma(N_2)$ by the additivity property of the signature. By obstruction theory, we show that N is almost parallelizable. N is certainly parallelizable over int $N_1 \cup$int N_2. The obstruction to extending a trivialization of the tangent bundle of $N-$point from the complement of a tubular neighborhood of ∂N_i in N over all of $N-$point is in

$$H^{q+1}(N - \text{point, int } N_1 \cup \text{int } N_2; \pi_q SO_{4k})$$

which is isomorphic to

$$H^{q+1}((\partial N_1 - \text{point}) \times (I, \partial I); \pi_q SO_{4k}) \cong H^q(\partial N_1 - \text{point}; \pi_q SO_{4k}) = :H^q$$

by excision. So the obstruction is certainly zero in case (i), and by the $(2k-2)$-connectedness of ∂N_1 and Poincaré duality in case (ii) except possibly if $q=2k-1$ or $2k$. We have $\pi_q SO_{4k} \cong \pi_q SO$ in these dimensions. We distinguish four cases. $k \equiv 0 \bmod 4: \pi_{2k-1}SO=\boldsymbol{Z}, \pi_{2k}SO=\boldsymbol{Z}_2 \Rightarrow H^{2k-1}=0$, $H^{2k}=0$ because of $(*)$. $k\equiv 1 \bmod 4: \pi_{2k-1}SO=\boldsymbol{Z}_2$, $\pi_{2k}SO=0 \Rightarrow H^{2k-1}=0$ because of $(*)$, $H^{2k}=0$. $k\equiv 2 \bmod 4: \pi_{2k-1}SO=\boldsymbol{Z}$, $\pi_{2k}SO=0 \Rightarrow H^{2k-1}=0$, $H^{2k}=0$. $k\equiv 3 \bmod 4: \pi_{2k-1}SO=0$, $\pi_{2k}SO=0 \Rightarrow H^{2k-1}=0$, $H^{2k}=0$.

Thus the obstruction vanishes in all cases, and N is almost parallelizable. But an almost parallelizable closed $(4k)$-manifold has signature divisible by $8\omega_{4k}$ ([**12**, p. 457], [**8**, p. 530]). So $\sigma(N_1)-\sigma(N_2)\equiv 0 \bmod 8\omega_{4k}$.

Conversely, assume $\sigma(N_1)\equiv\sigma(N_2) \bmod 8\omega_{4k}$. If B_i is a smooth ball in ∂N_i, then ∂N_1-B_1 and ∂N_2-B_2 are diffeomorphic by an orientation preserving diffeomorphism. We use this diffeomorphism to glue N_1 to N_2 along ∂N_1-B_1 and ∂N_2-B_2, and straighten the corners. The resulting smooth manifold P has boundary Σ a homotopy sphere such that $\partial N_1 \cong \partial N_2 \# \Sigma$. The obstructions to trivializing the tangent bundle of P vanish by the same computation as above, so P is parallelizable. But

$$\sigma(P) = \sigma(P \cup \text{cone over } \partial P) = \sigma(N_1) - \sigma(N_2) \equiv 0 \bmod 8\omega_{4k}.$$

Therefore Σ is a standard sphere, and ∂N_1 is orientation preservingly diffeomorphic to ∂N_2.

REMARK. If we allow $H_{2k-1}(\partial N_1)$ to have positive rank r in case (ii), the signature of N_1 still determines the smooth structure of ∂N_1. For, using [**16**, Theorem 1], one can show that in this case N_1 splits off r copies of $S^{2k}\times D^{2k}$. This generalizes Theorem 1, but does not generalize the Corollary, because if $H_{2k-1}(\partial N_1)$ has a free part, we do not know whether σ_1 is an invariant of the topological type of the embedding $\partial N_1 \subset S^{4k+1}$ (cf. [**7**, 5.1]).

THEOREM 2. *For* $k=3, 4, 5, 7, 15$, *there exist two* $(4k-1)$-*dimensional manifolds* M_1, M_2 *with the following properties*:

(i) M_1 *is homeomorphic to* M_2.

(ii) M_1 *is not diffeomorphic to* M_2.

(iii) $H_{2k-2}(M_i; Z)$ *and* $H_{2k-1}(M_i; Z)$ *are zero.*

(iv) M_1 *and* M_2 *are embeddable in* S^{4k+1} *with signature* σ_1 *equal to zero.*

This theorem shows that in general the signature of the embedding does not detect the smooth structure of the embedded manifold. Recall that condition (iii) was used to assure the topological invariance of the signature.

PROOF OF THEOREM 2. Our examples are homeomorphic to products of spheres, namely to $S^9\times S^2$, $S^{10}\times S^1$, $S^{13}\times S^2$, $S^{18}\times S^1$, $S^{17}\times S^{10}$, $S^{33}\times S^{26}$. Here, $k=3, 3, 4, 5, 7, 15$, respectively. The idea is to find an exotic homotopy n-sphere Σ^n which embeds in S^{4k+1} with trivial normal bundle and has the property that $\Sigma^n\times S^{4k-1-n}$ is not diffeomorphic to $S^n\times S^{4k-1-n}$. Then both these manifolds can be embedded with signature zero, namely in such a way that they bound $\Sigma^n\times D^{4k-n}$, and $S^n\times D^{4k-n}$, respectively.

Let Φ_n^m be the subgroup of Θ_n consisting of those homotopy n-spheres that embed in S^{n+m} with trivial normal bundle ($n\geqq 5$). It is well known that, for $m\geqq 2$, $\Sigma^n\times S^m$ is diffeomorphic to $S^n\times S^m$ if and only if $\Sigma^n\in\Phi_n^{m+1}$ ([**2**], [**8**]). For $m=1$, $\Sigma^n\times S^1$ and $S^n\times S^1$ are diffeomorphic if and only if Σ^n and S^n are diffeomorphic. It is clear from the definition of Φ_n^{m+2} that $\Sigma^n\times D^{m+1}$ embeds in S^{n+m+2} if and only if $\Sigma^n\in\Phi_n^{m+2}$.

LEMMA. *For* $(k, n)=(3, 9), (3, 10), (4, 13), (5, 18), (7, 17)$, *and* $(15, 33)$, Φ_n^{4k-n} *is a proper subgroup of* Φ_n^{4k+1-n}.

For any (k, n) mentioned in this Lemma (the proof of which we postpone), let Σ^n be a homotopy sphere in $\Phi_n^{4k+1-n}-\Phi_n^{4k-n}$. Then $\Sigma^n\times D^{4k-n}$ embeds in S^{4k+1}, and $M_1=\Sigma^n\times S^{4k-1-n}$ and $M_2=S^n\times S^{4k-1-n}$ are not diffeomorphic. After checking condition (iii), the proof of Theorem 2 is complete.

PROOF OF THE LEMMA. $(k, n)=(3, 9)$: $\Phi_9^4=\Theta_9\neq 0$ by [**3**], but $\Phi_9^3\neq\Theta_9$ according to [**10**, 7.4].

$(k, n)=(3, 10)$: By [**10**, 7.4] and the fact that $\Theta_{10}\simeq Z_6$, there is an exotic homotopy 10-sphere embeddable in S^{13}. Any embedding of a homotopy sphere in a sphere with codimension 3 has trivial normal bundle [**11**]. Therefore $\Phi_{10}^3\neq 0$. On the other hand, $\Phi_{10}^2=bP_{11}=0$.

$(k, n)=(4, 13)$: By [**3**], $\Phi_{13}^3=0$, $\Phi_{13}^4=\Theta_{13}\neq 0$.

$(k, n)=(5, 18)$: We will show $\Phi_{18}^3\neq 0$. This is sufficient because $\Phi_{18}^2=0$. As the Kervaire invariant of 18-manifolds is zero [**4**], and as codimension 3 embeddings of homotopy spheres have trivial normal bundle [**11**], by [**10**, 6.8] we have $\Theta_{18}/\Phi_{18}^3\simeq\mathrm{cok}(s)$ where s is the suspension

$$s:\pi_{18}(G_3, SO_3)\rightarrow\pi_{18}(G, SO).$$

Consider the following commutative diagram in which the rows are pieces of exact homotopy sequences of pairs:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{18}F_2 & \longrightarrow & \pi_{18}(F_2, SO_2) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow{\scriptstyle \cong} & & & & \\
 & & \pi_{18}G_3 & \longrightarrow & \pi_{18}(G_3, SO_3) & & & & \\
 & & \downarrow & & \downarrow{\scriptstyle s} & & & & \\
0 & \longrightarrow & \pi_{18}G & \longrightarrow & \pi_{18}(G, SO) & \longrightarrow & \pi_{17}SO & \xrightarrow{J} & \pi_{17}G
\end{array}$$

The stable J-homomorphism J is injective by [**1**, Theorem 1.3], so $\pi_{18}G \rightarrow \pi_{18}(G, SO)$ is an isomorphism. The isomorphism $\pi_{18}(F_2, SO_2) \rightarrow \pi_{18}(G_3, SO_3)$ is well known, cf. [**10**, 6.3]. Thus $\text{cok}(s) = \text{cok}(\pi_{18}F_2 \rightarrow \pi_{18}G)$. The direct summand Z_2 of $\pi_{18}F_2 \cong \pi_{20}S^2$, generated by $\eta_2 \circ \bar{\mu}_3$ [**15**, Theorem 12.8] under suspension maps onto the direct summand Z_2 of $\pi_{18}G \cong Z_8 \oplus Z_2$ which is generated by $\eta \circ \bar{\mu}$ [**15**, Theorem 12.22]. Therefore

$$\text{order } \text{cok}(s) < 16 = \text{order } \Theta_{18},$$

and $\Phi^3_{18} \neq 0$.

$(k, n) = (7, 17)$: By [**2**], $\Theta_{17}/\Phi^{11}_{17} \cong Z_2$, $\Theta_{17}/\Phi^{12}_{17} = 0$.

$(k, n) = (15, 33)$: By [**2**], $\Theta_{33}/\Phi^{27}_{33} \cong Z_2$, $\Theta_{33}/\Phi^{28}_{33} = 0$.

References

1. J. F. Adams, *On the groups $J(X)$*. IV, Topology **5** (1966), 21–71; correction, ibid. **7** (1968), 331. MR **33** #6628; MR **37** #5874.

2. P. Antonelli, *On stable diffeomorphism of exotic spheres in the metastable range*, Institute for Advanced Study, Princeton, N.J., 1970 (preprint).

3. F. Bandô and K. Katase, *A remark on the π-imbedding of homotopy spheres*, Proc. Japan Acad. **45** (1969), 443–445. MR **40** #6566.

4. E. H. Brown, Jr. and F. P. Peterson, *The Kervaire invariant of $(8k+2)$-manifolds*, Bull. Amer. Math. Soc. **71** (1965), 190–193. MR **30** #584.

5. R. De Sapio, *Differential structures on a product of spheres*, Comment. Math. Helv. **44** (1969), 61–69. MR **39** #4857.

6. A. Durfee, *Diffeomorphism classification of isolated hypersurface singularities*, Thesis, Cornell University, Ithaca, N.Y., 1971.

7. D. Erle, *Quadratische Formen als Invarianten von Einbettungen der Kodimension* 2, Topology **8** (1969), 99–114. MR **38** #6576.

8. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537. MR **26** #5584.

9. R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. **75** (1969), 742–749. MR **39** #3500.

10. J. Levine, *A classification of differentiable knots*, Ann. of Math. (2) **82** (1965), 15–50. MR **31** #5211.

11. W. S. Massey, *On the normal bundle of a sphere imbedded in Euclidean space*, Proc. Amer. Math. Soc. **10** (1959), 959–964. MR **22** #237.

12. J. W. Milnor and M. A. Kervaire, *Bernoulli numbers, homotopy groups, and a theorem of Rohlin*, Proc. Internat. Congress Math. (Edinburgh, 1958), Cambridge Univ. Press, New York, 1960, pp. 454–458. MR **22** #12531.

13. J. R. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554. MR **22** #12534.

14. R. E. Schultz, *Smooth structures on* $S^p \times S^q$, Ann. of Math. (2) **90** (1969), 187–198. MR **40** #3560.

15. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, no. 49, Princeton Univ. Press, Princeton, N.J., 1962. MR **26** #777.

16. C. T. C. Wall, *Classification of* $(n-1)$*-connected* $2n$*-manifolds*, Ann. of Math. (2) **75** (1962), 163–189. MR **26** #3071.

17. ———, *Classification problems in differential topology*. VI. *Classification of* $(s-1)$*-connected* $(2s+1)$*-manifolds*, Topology **6** (1967), 273–296. MR **35** #7343.

Mathematisches Institut der Universität Bonn, 53 Bonn, West Germany