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λ-EQUIVARIANT BORDISM OF MAPS

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ABSTRACT. It is shown that λ-equivariant maps may be classified up to bordism where the acting groups are abelian of odd order.

1. **Introduction.** R. E. Stong in [3] has shown the Conner and Floyd fixed point methods may be used to classify equivariant maps between closed manifolds with G-action up to bordism when the group acting is cyclic of prime order. In [1] it has been shown that these methods can also be used to solve the case for the group being abelian of odd order. The object here is to show that the results in [1] can be extended to classify λ-equivariant maps up to bordism where the acting groups are abelian of odd order.

2. **Definition of the λ-equivariant bordism groups.** Let $\lambda: H \to G$ be a continuous homomorphism between finite groups. A *λ-equivariant map of dimension* (m, n) is a triple $((M^m, \psi), (N^n, \theta), f)$ where M^m and N^n are compact manifolds of dimension m and n, $\psi: H \times M \to M$ is a differentiable H-action, $\theta: G \times N \to N$ is a differentiable G-action, and $f: (M, \psi) \to (N, \theta)$ is a differentiable map λ-equivariant with respect to the given actions; i.e. $f(\psi(h, x)) = \theta(\lambda(h), f(x))$. Two λ-equivariant maps of dimension (m, n), $((M, \psi), (N, \theta), f)$ and $((M', \psi'), (N', \theta'), f')$, are equivalent if there is a triple $((V, \Psi), (W, \Theta), F)$ where V and W are compact manifolds with boundary, M and M' being regularly imbedded submanifolds of ∂V with $\partial V = M \cup M'$, $M \cap M' = \varnothing$ and N and N' are similarly related in ∂W; Ψ is a differentiable H-action extending ψ and ψ', Θ is a differentiable G-action extending θ and θ'; and $F: (V, \Psi) \to (W, \Theta)$ is a differentiable λ-equivariant map extending f and f'.

The disjoint union of maps makes the set of equivalence classes into an abelian group (Z_2 vector space) denoted $\mathfrak{N}^{\lambda}_{m,n}$. In [3] Stong defines the concept of an $(\mathfrak{F}, \mathfrak{F}')$-free map of dimension (m, n) and the corresponding bordism group $\mathfrak{N}^{G}_{m,n}(\mathfrak{F}, \mathfrak{F}')$. The above definition agrees with that of Stong for the case of λ the identity map and, M^m and N^n both $(\mathfrak{All}, \varnothing)$-free, where $\mathfrak{All}$ is the collection of all subgroups of G and $\varnothing$ is the empty family. In this case the group is denoted $\mathfrak{N}^{G}_{m,n}$.

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3. **Computability.** There is the well-known result that for any pair of spaces (Y, B), $\mathfrak{N}_*(Y, B) \cong H_*(Y, B; Z_2) \otimes_{Z_2} \mathfrak{N}_*$. This motivates the following definition: A λ-equivariant bordism group is *computable* if it is isomorphic to a direct sum of ordinary unoriented bordism groups of some spaces.

The principal results of **[1]** show that for G abelian of odd order and $\mathfrak{F} \supset \mathfrak{F}'$ families adjacent with respect to K (see **[4]** for definitions), then there exist spaces $S_K(m, n-k)$ such that $\mathfrak{N}^G_{m,n}(\mathfrak{F}, \mathfrak{F}') \cong \bigoplus_{k=0}^n \mathfrak{N}_k(S_K(m, n-k))$. The space $S_K(m, n-k)$ is the union of a collection of spaces constructed out of the product of a fixed set of BO^G_{n-k}, a universal G/K-space described in **[2]**, and a space derived from a Pontrjagin-Thom construction. Next one supposes that $\mathfrak{F} \supset \mathfrak{F}'$ is any pair of families. Letting $\mathscr{C}$ be the collection of all subgroups in $\mathfrak{F} - \mathfrak{F}'$, and $\mathscr{S}(m, n-k)$ the disjoint union over $K \in \mathscr{C}$ of $S_K(m, n-k)$, it is shown that $\mathfrak{N}^G_{m,n}(\mathfrak{F}, \mathfrak{F}') \cong \bigoplus_{k=0}^n \mathfrak{N}_k(\mathscr{S}(m, n-k))$. Thus, for example, $\mathfrak{N}^G_{m,n}(\mathfrak{All}, \varnothing) = \mathfrak{N}^G_{m,n}$ is computable.

Now consider diagrams of the form:

$$\begin{tikzcd} M^m \arrow[r, "f"] \arrow[d, "d"'] & N^n \arrow[d, "e"] \\ X & Y \end{tikzcd}$$

where G acts on X, M^m, N^n, and Y, by σ, ψ, θ, and τ respectively; and d, f, and e are differentiable equivariant maps. Then there is the obvious extension of the definitions of $\mathfrak{N}^G_{m,n}$; requiring that there exist differentiable equivariant maps $D: V \to X$ and $E: W \to Y$ which restrict to d and d', and to e and e' respectively. The corresponding bordism group is denoted $\mathfrak{N}^G_{m,n}(X, \sigma; Y, \tau)$. The groups corresponding to the diagrams $X \xleftarrow{d} M^m \xrightarrow{f} N^n$ and $M^m \xrightarrow{f} N^n \xrightarrow{e} Y$ are denoted by $\mathfrak{N}^G_{m,n}(X, \sigma; -)$ and $\mathfrak{N}^G_{m,n}(-; Y, \tau)$ respectively. Note that $\mathfrak{N}^G_{m,n}(-; -) = \mathfrak{N}^G_{m,n}$.

It is immediate from the proofs in **[1]** that for G abelian of odd order, by including in the product forming $S(m, n-k)$ the spaces X and Y, $\mathfrak{N}^G_{m,n}(X, \sigma; Y, \tau)$ is computable. I.e., there exist spaces $\mathscr{S}(m, n-k; X, Y)$ such that $\mathfrak{N}^G_{m,n}(X, \sigma; Y, \tau) \cong \bigoplus_{k=0}^n \mathfrak{N}_k(\mathscr{S}(m, n-k; X, Y))$. Clearly $\mathfrak{N}^G_{m,n}(X, \sigma; -)$ and $\mathfrak{N}^G_{m,n}(-; Y, \tau)$ are also computable.

4. **The case for H and G abelian of odd order.**

LEMMA *If $\lambda: H \to G$ is a monic homomorphism between odd order abelian groups, then $\mathfrak{N}^\lambda_{m,n}$ is isomorphic to $\mathfrak{N}^G_{m,n}(G/H, \mu; -)$, with the obvious G-action, μ, on G/H.*

PROOF. If λ is monic then H is isomorphic to a subgroup of G. So without loss of generality one may assume that H is a subgroup of G. Let

$((M^m, \psi), (N^n, \theta), f)$ represent an element in $\mathfrak{N}^{\lambda}_{m,n}$. Stong [**4**, §4] defines the extensions $e(M)$, $e(\psi)$, and $e(f)$ to give a G-equivariant map $e(f)$: $e(M) \to N$. Let $\pi : e(M) \to G/H$ be defined by $\pi(g, x) = gH$. Then $G/H \xleftarrow{\pi} e(M) \xrightarrow{e(f)} N$ is a G-equivariant diagram representing an element in $\mathfrak{N}^{G}_{m,n}(G/H, \mu; -)$.

Now suppose that $G/H \xleftarrow{p} \bar{M} \xrightarrow{\bar{f}} N$ represents an element in $\mathfrak{N}^{G}_{m,n}(G/H, \mu; -)$. Let $M = p^{-1}(H/H)$, and $\psi = \bar{\psi}|_M$. Let $i : M \to \bar{M}$ be the inclusion map and $f = \bar{f} \circ i$. M is H-invariant and f is λ-equivariant. Thus $f : M \to N$ represents an element of $\mathfrak{N}^{\lambda}_{m,n}$. One notes that $e(M) = \bar{M}$ and $e(f) = \bar{f}$. So one associates to $((M, \psi), (N, \theta), f)$ the bordism class $((e(M), e(\psi)), (N, \theta), f; (G/H, \mu), p)$. Thus $\mathfrak{N}^{\lambda}_{m,n}$ is isomorphic to $\mathfrak{N}^{G}_{m,n}(G/H, \mu; -)$.

PROPOSITION. *Suppose that $\lambda : H \to G$ is a homomorphism between odd order abelian groups. Then $\mathfrak{N}^{\lambda}_{m,n}$ is computable.*

PROOF. Let $((M, \psi), (N, \theta), f)$ represent an element in $\mathfrak{N}^{\lambda}_{m,n}$. Since G is abelian one has that $H \times G$ acts on N by $\zeta((h, g), n) = \theta(\lambda(h)g, n)$. Since H is abelian the elements $(h, \lambda(h^{-1}))$ where $h \in H$ form a subgroup of $H \times G$ which fixes N. Denote this subgroup by F. Let $\bar{\lambda} : H \to H \times G$ be the monic homomorphism defined by $h \mapsto (h, 1)$. Since f is λ-equivariant one has that f is $\bar{\lambda}$-equivariant. Thus $((M, \psi), (N, \zeta), f)$ represents an element of $\mathfrak{N}^{\bar{\lambda}}_{m,n}$ where F fixes N.

One notes that if an element in $\mathfrak{N}^{\bar{\lambda}}_{m,n}$ has a representative of the form $((\bar{M}, \bar{\psi}), (\bar{N}, \bar{\zeta}), \bar{f})$ where $\bar{N}$ is fixed by F, then it represents an element of $\mathfrak{N}^{\lambda}_{m,n}$ where the G-action on $\bar{N}$ is defined by $\psi(g, n) = \bar{\zeta}((1, g), n)$. By the lemma $\mathfrak{N}^{\bar{\lambda}}_{m,n}$ is isomorphic to $\mathfrak{N}^{H \times G}_{m,n}(G, \mu; -)$. Thus $\mathfrak{N}^{\lambda}_{m,n}$ is isomorphic to the subgroup of $\mathfrak{N}^{H \times G}_{m,n}(G, \mu; -)$ of those elements with a representative (M, N, f) where F fixes N. Let $\mathscr{C}$ be the collection of all subgroups of $H \times G$ which contain F. Let $\mathscr{S}(m, n-k; G)$ be the disjoint union $\bigcup_{K \in \mathscr{C}} S_K(m, n-k; G, -)$ where $S_K(m, n-k; G, -)$ is the space constructed from [**1**]. Then one observes that $\mathfrak{N}^{\lambda}_{m,n}$ is isomorphic to $\bigoplus_{k=0}^{n} \mathfrak{N}_k(\mathscr{S}(m, n-k; G))$.

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