FURTHER COMMENTS ON THE CONTINUITY OF DISTRIBUTION FUNCTIONS OBTAINED BY SUPERPOSITION

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Abstract. Let \( \{X(t)\} \) be a differential process with discontinuous distributions and \( Y \) a nonnegative random variable independent of the process. The superposition \( X(Y) \) has a continuous probability distribution if and only if the process has nonzero trend term and \( Y \) has continuous distribution. The nature of discontinuities of the probability distribution of the superposition is indicated.

We continue the notation and terminology of [1] and [3]. Let \( \{X(t)|t \in [0, \infty)\} \) be a differential process (homogeneous process) with discontinuous distributions. Then

\[
X(t) = \tau_X t + X^*(t),
\]

where

\[
f_{X^*(t)}(u) = Ee^{iuX^*(t)} = \exp\left\{ t \int_{-\infty}^{\infty} (e^{iu} - 1) \, dM_X(x) \right\}
\]

and the Lévy spectral function satisfies

\[
\int_{-\infty}^{\infty} dM_X(x) = \int_{-\infty}^{0} dM_X(x) + \int_{0}^{\infty} dM_X(x) = \mu + \lambda < \infty.
\]

\( \tau_X \) is the trend term of the process. Let \( Y \geq 0 \) be independent of the \( \{X(t)\} \) process and consider the superposition \( X(Y) \). We shall show that \( X(Y) \) has continuous distribution if and only if the process has nonzero trend term and \( Y \) has continuous distribution. The nature of discontinuities will be indicated.

Lemma 1. Let \( \{X^*(t)\} \) be a differential process with discontinuous distributions and no trend term. Then

\[
\text{Cont } F_{X^*(t)}(\cdot) = \text{Cont } F_{X^*(t)}(\cdot), \quad \forall t > 0.
\]

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Proof. Reviewing the argument of Theorem 2 in [1], we see that \( F_{X^1(t)}(\cdot) \) is the distribution function of the random sum \( Z(t) = X_1 + \cdots + X_{Y(t)} \), where \( X_1, X_2, \ldots \) are independent with common distribution

\[
G(x) = M_X(x)(\mu + \lambda), \quad x < 0,
\]

\[
= \mu(\mu + \lambda), \quad x = 0,
\]

\[
= (\mu + \lambda + M_X(x))(\mu + \lambda), \quad x > 0,
\]

and \( \mathcal{L}(Y(t)) = \mathcal{P}(i(\mu + \lambda)). \) Thus \( f_{X^1(t)}(u) = \sum_{k=0}^{\infty} (f_{X^1}(u))^k P[Y(t) = k] \). Let \( j_{X^1(t)}(a) = F_{X^1(t)}(a) - F_{X^1(t)}(a-) \) be the jump of \( F_{X^1(t)}(\cdot) \) at \( a \).

Applying Theorem 3.2.3 of [2], Fubini's Theorem, and the Lebesgue Dominated Convergence Theorem, we obtain

\[
j_{X^1(t)}(a) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-iuj_{X^1(t)}(u)} du
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} e^{-iu} \sum_{k=0}^{\infty} (f_{X^1}(u))^k P[Y(t) = k] du
\]

\[
= \lim_{T \to \infty} \sum_{k=0}^{\infty} \frac{1}{2T} \int_{-T}^{T} e^{-iu} (f_{X^1}(u))^k P[Y(t) = k]
\]

\[
= e^{-i(\mu + \lambda)} + \sum_{k=1}^{\infty} j_{X^1(t)}(a) P[Y(t) = k], \quad a = 0,
\]

\[
= \sum_{k=1}^{\infty} j_{X^1(t)}(a) P[Y(t) = k], \quad a \neq 0.
\]

Thus \( F_{X^1(t)}(\cdot) \) has a jump at \( a \) if and only if some \( F_{X^1(t)}(\cdot) \) has a jump at \( a \); i.e.,

\[
\overline{\text{Cont} F_{X^1(t)}(\cdot)} = \{0\} \cup \overline{\text{Cont}} F_{X^1(\cdot)} \cup \cdots, \quad \forall t > 0,
\]

and the lemma is proved. \( \square \)

Note that \( j_{X^1(t)}(a) = \sum_{k=0}^{\infty} j_{X^1(t)}(a) P[Y(t) = k] \) and the Helly-Bray Theorem imply that \( j_{X^1(t)}(a) \) is continuous for \( a \) fixed.

Lemma 2. Let \( \{X(t) = \tau_X t + X^*(t)\} \) be a differential process with discontinuous distributions and nonzero trend term. Then for each fixed \( a \), \( \{t | j_{X^1(t)}(a) \neq 0\} \) is at most countable.

Proof. Applying Lemma 1, we note that \( a \notin \text{Cont} F_{X^1(t)}(\cdot) \) if and only if \( a - \tau_X t \notin \text{Cont} F_{X(t)}(\cdot) = \text{Cont} F_{X^1(t)}(\cdot) \). Thus \( j_{X^1(t)}(a) \neq 0 \) if and only if \( t = (a - x)/\tau_X \) for some \( x \notin \text{Cont} F_{X^1(t)}(\cdot) \). \( \square \)
Theorem 1. Let \( Y \geq 0 \) be independent of the differential process \( \{X(t)\} \). Then for each fixed \( a \)

\[
j_X(Y)(a) = \int_0^\infty j_{X(0)}(a) \, dF_Y(t).
\]

Proof. Using the arguments of Lemma 1, we obtain

\[
j_X(Y)(a) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T e^{-iau} f_{X(Y)}(u) \, du
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T e^{-iau} f_{X(0)}(u) \, dF_Y(t) \, du
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T f_{X(0)}(u) \, dF_Y(t)
\]

\[
= \int_0^\infty j_{X(0)}(a) \, dF_Y(t).
\]

Corollary 1. Let \( \{X(t)\} \) be a differential process with discontinuous distributions and nonzero trend term. Suppose \( Y \geq 0 \) is independent of the process and has continuous distribution. Then the superposition \( X(Y) \) has continuous distribution.

Proof. The integrand in (1) vanishes a.e. by Lemma 2 and \( Y \) has no point masses.

Corollary 2. Let \( \{X(t)\} \) be a differential process with discontinuous distributions and nonzero trend term. Suppose \( Y \geq 0 \) is independent of the process and has a discontinuous distribution. Then \( X(Y) \) has a discontinuous distribution with jumps occurring at precisely those points of the form \( a = t_0 + \xi \), where \( t_0 \notin \text{Cont } F_Y(\cdot) \) and \( \xi \notin \text{Cont } F_{X^*(t)}(\cdot) \).

Proof. The indicated points are precisely those where a positive value of the integrand in (1) coincides with a point mass of \( Y \).

Corollary 3. Let \( \{X^*(t)\} \) be a differential process with discontinuous distributions and no trend term. Suppose \( Y \geq 0 \) is independent of the process and \( P[Y=0]<1 \). Then \( X^*(Y) \) has discontinuous distribution and

\[
\text{Cont } F_{X^*(Y)}(\cdot) = \text{Cont } F_{X^*(t)}(\cdot).
\]

Proof. The integrand in (1) vanishes if \( a \in \text{Cont } F_{X^*(t)}(\cdot) \). If \( a \notin \text{Cont } F_{X^*(t)}(\cdot) \), then (1) and the observation that \( j_{X^*(t)}(a) \) is continuous and positive imply that \( j_{X^*(t)}(a) > 0 \).

We also note that (1) immediately yields Corollary 1A of [1].
REFERENCES

