PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

TOPOLOGICAL PROPERTIES OF THE EFFICIENT POINT SET

BEZALEL PELEG

ABSTRACT. Let Y be a closed and convex subset of a Euclidean space. We prove that the set of efficient points of Y, $M(Y)$, is contractible. Furthermore, if $M(Y)$ is closed (compact) then it is a retract of a convex closed (compact) set. Our proof relies on the Arrow-Barankin-Blackwell Theorem. A new proof is supplied for that theorem.

1. **Introduction.** The study of efficient points of convex sets is expounded by many writers (see, e.g., [3], [**2**, pp. 306–310], [**6**, §12.3]). In particular, topological properties of the efficient point set are investigated in [**3**, pp. 73–78]. This paper is a further contribution on this topic: In §4 we prove that the set of efficient points $M(Y)$ of a closed and convex subset Y of a Euclidean space is contractible. Furthermore, if $M(Y)$ is closed (compact) then it is a retract of a convex closed (compact) set.

Our proofs make use of the Arrow-Barankin-Blackwell Theorem [**1**, Theorem 1]. This theorem is generalized in infinite-dimensional spaces in [**9**], [**7**], [**8**], [**5**], and [**4**]. In §3 we offer a new proof of the Arrow-Barankin-Blackwell Theorem. Our proof is advantageous over the original one in two respects: It is a "constructive" proof, unlike that of Arrow, Barankin, and Blackwell. Furthermore, it is easier to generalize to infinite-dimensional spaces (see [**7**]).

I am indebted to M. Perles and M. Yaari for many helpful discussions.

2. **Preliminaries.** Let E^n be the n-dimensional Euclidean space. If $x, y \in E^n$, then we write $x \geqq y$ if $x_i \geqq y_i$ for $i=1, \cdots, n$. $x>y$ if $x \geqq y$ and $x \neq y$. $x \gg y$ if $x_i > y_i$ for $i=1, \cdots, n$. We denote by E_+^n the nonnegative cone of E^n, i.e., $E_+^n = \{x | x \in E^n \text{ and } x \geqq 0\}$. The scalar product of two members x and y of E^n is denoted by $x \cdot y = \sum_{i=1}^n x_i y_i$. The norm of a member x of E^n is denoted by $\|x\| = (x \cdot x)^{1/2}$. $u^{(i)}$, $i=1, \cdots, n$, will denote the ith unit vector of E^n.

3. **An alternative proof of the Arrow-Barankin-Blackwell Theorem.** Let Y be subset of E^n. A point $e \in Y$ is an *efficient point* of Y if there exists no

Received by the editors December 9, 1971 and, in revised form, January 31, 1972.
AMS 1970 *subject classifications*. Primary 90A99; Secondary 90D12, 54F05.

$y \in Y$ such that $y > e$. $r \in Y$ is a *regular efficient point* of Y if there exists a vector $p \in E^n$, $p \gg 0$, such that $p \cdot r \geqq p \cdot y$ for all $y \in Y$. Clearly, a regular efficient point of Y is an efficient point of Y. Let

$$M(Y) = \{e \mid e \text{ is an efficient point of } Y\}. \tag{3.1}$$

THEOREM 3.1 (ARROW, BARANKIN, AND BLACKWELL [1]). *Let Y be a closed and convex subset of E^n. The regular efficient points of Y are dense in $M(Y)$.*

PROOF. Let $e \in M(Y)$ and let $Y^* = \{y \mid y \in Y \text{ and } \|y-e\| \leqq 1\}$. If r^* is a regular efficient point of Y^* and $\|r^*-e\| < 1$ then r^* is a regular efficient point of Y. To see this let $p \gg 0$ satisfy $p \cdot r^* \geqq p \cdot y^*$ for all $y^* \in Y^*$. Let $y \in Y$. For $t > 0$ sufficiently small, $ty + (1-t)r^* \in Y^*$. Hence, $p \cdot r^* \geqq t \cdot p \cdot y + (1-t) p \cdot r^*$. Thus, $p \cdot r^* \geqq p \cdot y$. Thus, it is sufficient to prove that e is the limit of a sequence of regular efficient points of Y^*. But Y^* is compact. Hence, we may assume that Y is *compact*. We may assume further that $Y \subset E^n_+$. Let $C = \max\{\|y\| \mid y \in Y\}$. For each k, $k = 1, 2, \cdots$, let

$$Y^{(k)} = \{y \mid y \in Y \text{ and } y_i \geqq e_i - 1/k,\ i = 1, \cdots, n\}, \tag{3.2}$$

$$v_k(x) = \min(x_i - e_i + 1/k,\ 1 \leqq i \leqq n), \qquad x \in E^n, \tag{3.3}$$

$$w_k(x) = \sum_{i=1}^{n} x_i / n(k+1)C, \qquad x \in E^n, \tag{3.4}$$

and

$$u_k(x) = v_k(x) + w_k(x). \tag{3.5}$$

Let $r^{(k)}$ be a point where u_k attains its maximum in $Y^{(k)}$. u_k is concave; hence, the set

$$Z = \{z \mid z \in E^n \text{ and } u_k(z) > u_k(r^{(k)})\} \tag{3.6}$$

is convex. $Z \cap Y^{(k)} = \varnothing$. Hence, there exists a $p \in E^n$ such that

$$p \cdot z \geqq p \cdot y \quad \text{for all } z \in Z \text{ and } y \in Y^{(k)}. \tag{3.7}$$

By (3.3), (3.4), and (3.5), u_k is increasing, i.e., $x > y$ implies that $u_k(x) > u_k(y)$. Hence, it follows from (3.6) and (3.7) that $p \gg 0$. Furthermore,

$$p \cdot r^{(k)} \geqq p \cdot y \quad \text{for all } y \in Y^{(k)}. \tag{3.8}$$

Since $1/k \leqq u_k(e) \leqq u_k(r^{(k)})$, it follows from (3.4) that

$$r_i^{(k)} > e_i - 1/k, \qquad i = 1, \cdots, n. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$p \cdot r^{(k)} \geqq p \cdot y \quad \text{for all } y \in Y. \tag{3.10}$$

Thus, $r^{(k)}$ is a regular efficient point of Y. Since $r^{(k)} \in Y^{(k)}$, $k=1, 2, \cdots$, and e is efficient, $e=\lim_{k\to\infty} r^{(k)}$.

4. **A proof that the set of efficient points is contractible.** Let Y be a closed and convex subset of E^n and let $M(Y)\neq\varnothing$ (see (3.1)).

LEMMA 4.1. *There exist a vector $p \gg 0$ and a real number v such that $p\cdot y \leqq v$ for all $y \in Y$.*

PROOF. By Theorem 3.1 there exists a regular efficient point of Y.

COROLLARY 4.2. *For each $x \in E^n$ the set $\{y|y \in Y$ and $y\geqq x\}$ is compact.*

LEMMA 4.3. *Let $Y^*=\{x|$ there exists $y \in Y$ such that $y\geqq x\}$. Then Y^* is convex and closed and $M(Y)=M(Y^*)$.*

PROOF. It is clear that Y^* is convex and that $M(Y)=M(Y^*)$. To see that Y^* is closed let $x=\lim_{k\to\infty} x^{(k)}$, $x^{(k)} \in Y^*$, $k=1, 2, \cdots$. There exist $y^{(k)} \in Y$, $y^{(k)}\geqq x^{(k)}$, $k=1, 2, \cdots$. By Corollary 4.2 the sequence $(y^{(k)})$ is bounded. Hence, we may assume that there exists a vector y such that $y=\lim_{k\to\infty} y^{(k)}$. Clearly, $y \in Y$ and $y\geqq x$.

By Lemma 4.3 we may assume henceforth that $Y=Y^*$.

COROLLARY 4.4. *There exist points a, $b \in Y$ such that $b\gg a$.*

For $y \in Y$ we define

$$G(y) = \{x \mid x \in Y \text{ and } x \geqq y\}. \tag{4.1}$$

$G(y)$ is convex and compact. Also, G is an upper semicontinuous function of y.

LEMMA 4.5. *Let $y \in Y$. If there exists a $z \in Y$ such that $z\gg y$ then G is lower semicontinuous at y.*

PROOF. Let $y=\lim_{k\to\infty} y^{(k)}$ and let $x \in G(y)$. Let $1>t>0$.

$$x(t) = tz + (1-t)x \gg y.$$

Hence, there exists a natural number $k(t)$ such that $x(t) \in G(y^{(k)})$ for $k\geqq k(t)$. Since $\lim_{t\to 0} x(t)=x$, the lemma follows.

We recall that a topological space is *contractible* if its identity map is homotopic to a constant.

THEOREM 4.6. *$M(Y)$ is contractible.*

PROOF. Let p and v be as in Lemma 4.1. Let $w=\min_{1\leqq i\leqq n} p_i$. For $y \in Y$ let $f(y)=y+(v-p\cdot y)\sum_{i=1}^n u^{(i)}/w$. f is a continuous function of y.

Furthermore,

$$f(y) \geqq x \quad \text{for all } x \in G(y) \quad \text{(see (4.1))}. \tag{4.2}$$

For $y \in Y$ let $g(y) \in G(y)$ be the point defined by $\|g(y)-f(y)\| \leqq \|x-f(y)\|$ for all $x \in G(y)$. $g(y)$ is well defined. By (4.2), $g(y) \in M(Y)$. Now let $a \in Y$ be a point for which there exists a $b \in Y$ such that $b \gg a$ (see Corollary 4.4). For $e \in M(Y)$ and $0 \leqq t \leqq 1$ let $h(e, t)=g((1-t)e+ta)$. $h(e, 0)=e$ and $h(e, 1)=g(a)$ for all $e \in M(Y)$. Furthermore, h is a continuous function of both e and t. For let $t=\lim_{k\to\infty} t^{(k)}$ and $e=\lim_{k\to\infty} e^{(k)}$. If $t=0$ then

$$\lim_{k\to\infty}(1 - t^{(k)})e^{(k)} + t^{(k)}a = e.$$

$$h(e^{(k)}, t^{(k)}) \geqq (1 - t^{(k)})e^{(k)} + t^{(k)}a. \tag{4.3}$$

By Corollary 4.2 the sequence $(h(e^{(k)}, t^{(k)}))$ is bounded. Hence, it follows from (4.3) and from $e \in M(Y)$ that $\lim_{k\to\infty} h(e^{(k)}, t^{(k)})=e$. If $t>0$ then

$$(1 - t)e + tb \gg (1 - t)e + ta = \lim_{k\to\infty}(1 - t^{(k)})e^{(k)} + t^{(k)}a.$$

Hence, by Lemma 4.5, G is lower semicontinuous at $(1-t)e+ta$. Therefore, g is continuous at $(1-t)e+ta=y$. For assume, on the contrary, that

$$y = \lim_{k\to\infty} y^{(k)} \quad \text{and} \quad \lim_{k\to\infty} g(y^{(k)}) = z \neq g(y). \tag{4.4}$$

Then

$$\|g(y) - f(y)\| < \|z - f(y)\|. \tag{4.5}$$

Furthermore, there exist $x^{(k)} \in G(y^{(k)})$, $k=1, 2, \cdots$, such that

$$\lim_{k\to\infty} x^{(k)} = g(y). \tag{4.6}$$

It follows from (4.4)–(4.6) that there exists a k such that

$$\|x^{(k)} - f(y^{(k)})\| < \|g(y^{(k)}) - f(y^{(k)})\|,$$

which is impossible. The continuity of h at (e, t) follows now from the continuity of g at $(1-t)e+ta$.

We recall that a subset A of a topological space T is a *retract* of T if there exists a continuous function $r: T \to A$ such that $r(a)=a$ for all $a \in A$.

THEOREM 4.7. *If $M(Y)$ is closed then it is a retract of a closed and convex set.*

PROOF. For $y \in Y$ let $d(y, M(Y))$ be the distance between y and $M(Y)$. Let

$$t(y) = d(y, M(Y))/(1 + d(y, M(Y))). \tag{4.7}$$

Then $t(y)$ is a continuous function of y and $y \in M(Y)$ if and only if $t(y)=0$. Using the notation of the proof of Theorem 4.6 we define, for $y \in Y$,

$$h(y) = g((1 - t(y))y + t(y)a).$$

Then $h(y) \in M(Y)$ and $h(e)=e$ for $e \in M(Y)$. Furthermore, it follows from the definition of $t(y)$ and Lemma 4.5 that h is continuous. Hence, h is a retraction of Y on $M(Y)$.

THEOREM 4.8. *If $M(Y)$ is compact then it is a retract of a compact and convex set.*

PROOF. Choose $a \in E^n$ for which there exists $e \in M(Y)$ such that $e \gg a$. Let Y_1 be the convex hull of $M(Y) \cup \{a\}$. Then Y_1 is convex and compact and $M(Y_1)=M(Y)$. Let $q \in E^n$ satisfy $q \geqq x$ for all $x \in Y_1$. For $y \in Y_1$ let $g(y) \in G(y)=\{x | x \geqq y \text{ and } x \in Y_1\}$ be defined by

$$\|g(y) - q\| \leqq \|x - q\| \quad \text{for all } x \in G(y).$$

Let further $t(y)$ be defined by (4.7). Define now, for $y \in Y_1$,

$$h(y) = g((1 - t(y))y + t(y)a).$$

Then h is a retraction of Y_1 on $M(Y)$.

REMARK 4.9. If Y is polyhedral or strictly convex then $M(Y)$ is closed. However, Y may be compact without $M(Y)$ being closed.

5. **Concluding remarks.** Let Y be a closed and convex subset of a Euclidean space. Consider Y as a technology given in the flow version (see [**6**, §12]). By Theorem 4.6, $M(Y)$ is contractible; hence, in particular, it is arcwise connected. Thus, it is possible to move from one efficient process to another via $M(Y)$ in a continuous manner. This result may be useful for economic planning.

If Y is compact and polyhedral then, by Theorem 4.8, $M(Y)$ is a retract of a convex compact set. Therefore, every continuous function $f: M(Y) \to M(Y)$ has a fixed point. This last fact may prove useful in game theory and mathematical economics investigations.

REFERENCES

1. K. J. Arrow, E. W. Barankin and D. Blackwell, *Admissible points of convex sets*, Contributions to the Theory of Games, vol. 2, Ann. of Math. Studies, no. 28, Princeton Univ. Press, Princeton, N.J., 1953, pp. 87–91. MR **14**, 998.

2. D. Gale, *The theory of linear economic models*, McGraw-Hill, New York, 1960. MR **22** #6599.

3. T. C. Koopmans, *Analysis of production as an efficient combination of activities*, Activity Analysis of Production and Allocation, Wiley, New York; Chapman and Hall, London, 1951. MR **13**, 670.

4. M. Kurz and M. Majumdar, *Efficiency prices in infinite dimensional spaces: A synthesis* (to appear).

5. M. Majumdar, *Some approximation theorems on efficiency prices for infinite programs*, J. Economic Theory **2** (1970), 399–410.

6. H. Nikaido, *Convex structures and economic theory*, Academic Press, New York, 1968.

7. B. Peleg, *Efficiency prices for optimal consumption plans*, J. Math. Anal. Appl. **29** (1970), 83–90. MR **41** #5034.

8. ———, *Efficiency prices for optimal consumption plans*. II, Israel J. Math. **9** (1971), 222–234.

9. R. Radner, *A note on maximal points of convex sets in l_∞*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. 1: Statistics, Univ. of California Press, Berkeley, Calif., 1967, pp. 351–354. MR **35** #7111.

INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, ISRAEL