LOCALLY AFFINE RING EXTENSIONS
OF A NOETHERIAN DOMAIN

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Abstract. If $A \subseteq R$ are integral domains with $A$ noetherian, it is shown that if $R$ is contained in an affine ring over $A$ and if for each maximal ideal $P$ of $A$ with $S = A\setminus P$, $R_S$ is an affine ring over $A_P$, then $R$ itself is affine over $A$.

If $A \subseteq R$ are (commutative) integral domains and if for each prime ideal $P$ of $A$ with $S = A\setminus P$, $R_S$ is a finitely generated ring extension of $A_P$, then we say that $R$ is locally affine over $A$. If for each prime $P$ of $A$, $R_S$ is a polynomial ring over $A_P$, then $R$ is said to be locally a polynomial ring over $A$. Even for $A = \mathbb{Z}$, the ring of integers, it can happen that $A^*$ is locally affine over $A$ (and even locally polynomial over $A$), but yet $R$ is not finitely generated over $A$. Eakin and Silver in [1] give the following example. Let $\{p_i\}_{i=1}^n$ be the set of prime integers in $\mathbb{Z}$ and let $X$ be an indeterminate. Then $R = \mathbb{Z}[\{X/p_i\}_{i=1}^n]$ is locally a polynomial ring in one variable over $\mathbb{Z}$, but $R$ is not finitely generated over $\mathbb{Z}$. The question thus naturally arises of what additional conditions on $R$, locally affine over $A$, will imply that $R$ is finitely generated over $A$. Eakin and Silver [1] consider the question of whether $R$ locally a polynomial ring over $A$ and contained in an affine ring over $A$ imply $R$ is finitely generated over $A$. They prove this to be the case when $A$ is a Krull domain, when $A$ is a pseudo-geometric domain, or when $A$ is a 1-dimensional noetherian domain, and raise the question of whether in general $R$ locally polynomial over a noetherian domain $A$ and contained in an affine ring over $A$ imply $R$ is an affine ring over $A$. That the answer is yes is a consequence of the following.

Theorem. Let $A \subseteq R$ be integral domains with $A$ noetherian and $R$ locally affine over $A$. If $R$ is contained in an affine ring over $R$, then $R$ itself is affine over $A$.

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In proving this, our main theorem, we will make use of some properties of the representation of a noetherian integral domain $D$ in the form $D = \bigcap_a \{ D_{P_a} \mid P_a \text{ is an associated prime of a principal ideal of } D \}$. In general, if $\{V_a\}$ is a collection of integral domains with quotient field $K$, then $\{V_a\}$ is said to have finite character if each nonzero element of $K$ is a unit in all but finitely many of the $V_a$. If $D = \bigcap_a V_a$ also has quotient field $K$ and $\{V_a\}$ has finite character, then the representation $D = \bigcap_a V_a$ is said to be locally finite [3, p. 76]. Note that $D = \bigcap_a V_a$ is locally finite if and only if each nonzero element of $D$ is a unit in all but finitely many of the $V_a$. For an arbitrary integral domain $D$, it is well known that if $\{P_a\}$ is a collection of prime ideals of $D$ such that each associated prime of a principal ideal of $D$ is contained in some $P_a$, then $D = \bigcap_a D_{P_a}$ (see [4, p. 118] or [3, p. 34]). When $D$ is noetherian one has, moreover, the following.

**Lemma 1.** If $D$ is a noetherian domain, then the representation $D = \bigcap_a \{ D_{P_a} \mid P_a \text{ is an associated prime of a principal ideal of } D \}$ is locally finite.

**Proof.** Since $D$ is noetherian, if $P$ is an associated prime of a principal ideal of $D$, then $P$ is an associated prime of any nonzero $y \in P$. Thus, nonzero elements of $D$ are contained in only finitely many of the $P_a$ and the lemma follows.

**Lemma 2.** Let $D$ be a noetherian domain, let $\{P_a\}$ be the set of associated primes of principal ideals of $D$, and let $S$ be a multiplicative system in $D$. Then $D_S = \bigcap_a \{ D_{P_a} \mid S \cap P_a = \emptyset \}$.

**Proof.** Each associated prime of a principal ideal in $D_S$ is of the form $P_a \cap S = \emptyset$. Since for $P_a \cap S = \emptyset$, $D_{P_a} = (D_S)_{P_a} D_S$, the result follows.

We note the following immediate consequence of Lemma 2.

**Lemma 3.** Let $D \subseteq R$ be integral domains with $D$ noetherian and $R \subseteq D[1/f]$ for some nonzero $f \in D$. Let $\{P_a\}$ be the set of associated primes of principal ideals in $D$ and let $P_1, \ldots, P_n$ be the $P_a$ which contain $f$. If $s$ is a nonzero element of $D$ such that $R \not\subseteq D_{P_i}$ implies $s \in P_i$, then $R \subseteq D[1/s]$.

We shall also make use of the following standard lemma.

**Lemma 4.** Let $A$ be a commutative ring with identity and let $R$ be an $A$-algebra. If for each maximal ideal $P$ of $A$ there exists $s \in A \setminus P$ such that $R[1/s] = R \otimes_A A[1/s]$ is a finitely generated $A[1/s]$-algebra, then $R$ is a finitely generated $A$-algebra.

**Proof.** Let $\{P_\beta\}$ be the set of maximal ideals of $A$ and let $s_\beta \in A \setminus P_\beta$ be such that $R[1/s_\beta]$ is a finitely generated $A[1/s_\beta]$-algebra. Then $(s_\beta) = A$, so there exist $s_1, \ldots, s_n \in s_\beta$ such that $(s_1, \ldots, s_n) = A$. If $N_1$ is a finite
subset of $R$ such that $R[1/s_i] = A[1/s_i, N_i]$, then $N = N_1 \cup \cdots \cup N_n$ is such that $R = A[N]$.

**Proof of the Theorem.** Let $P$ be a maximal ideal of $A$, let $S = A \setminus P$, and let $x_1, \ldots, x_n \in R$ be such that $A_S[x_1, \ldots, x_n] = R_S$. By Lemma 4, it will suffice to show for some $s \in S$ that $R[1/s] = A[1/s, x_1, \ldots, x_n]$. Let $D = A[x_1, \ldots, x_n]$. Then $D \subset R$ and $R$ is contained in the quotient field of $D$. By assumption, $R$ is contained in an affine ring over $A$, so we have $R \subset D[\xi_1, \ldots, \xi_m]$. By taking a residue class ring of $D[\xi_1, \ldots, \xi_m]$ modulo a prime ideal lying over $(0)$ in $R$, we may assume that $D[\xi_1, \ldots, \xi_m]$ is an integral domain. Applying the normalization lemma of [4, p. 45], we may reduce modulo a suitable prime ideal to the case where $\xi_i$ are algebraic over $D$. Finally, writing the $\xi_i$ with denominators in $D$ and using the fact that $R$ is contained in the quotient field of $D$, we get a nonzero $f \in D$ such that $R \subset D[1/f]$. (A reduction to this case is also given by Eakin and Silver [1].) Since $A$ is noetherian, $D = A[x_1, \ldots, x_n]$ is noetherian. Let $\{P_a\}$ be the set of associated primes of principal ideals in $D$. Note that if $f \notin P_a$, then $D[1/f] \subset D_{P_a}$, so $R \subset D_{P_a}$. Let $P_1, \ldots, P_m$ be the $P_a$ which contain $f$. If $R \subset D_{P_i}$, then $P_i \cap A \not\subset P$; for $R \subset A_S[x_1, \ldots, x_n]$ and $P_i \cap A \subset P$ implies $A_S[x_1, \ldots, x_n] \subset D_{P_i}$. Hence we can choose $s \in A \setminus P$ such that $R \subset D_{P_i}$ implies $s \in P_i$. By Lemma 3, $R \subset D[1/s] = A[1/s, x_1, \ldots, x_n]$, which completes the proof of the Theorem.

**Remark.** It would be interesting to have more general conditions on $A$ and $R$ in order that $R$ locally affine over $A$ implies that $R$ is finitely generated over $A$. In our proof, the fact that $A$ is noetherian was used to insure that if $D$ is a domain finitely generated over $A$ and if $\{P_a\}$ is the set of associated primes of principal ideals of $D$, then the representation $D = \cap_a D_{P_a}$ is locally finite and for any nonzero $s \in D$, $D[1/s] = \cap_a (D_{P_a}[s] \not\subset P_a)$. Perhaps these properties hold for $A$ in a larger class of integral domains, thus allowing generalization of the above theorem by weakening the noetherian hypothesis on $A$. It is not true in general, however, for integral domains $A \subset R$ with $R$ locally affine over $A$ and contained in an affine ring over $A$ that necessarily $R$ is finitely generated over $A$. The following example illustrates this.

**Example.** Let $A^* \subset R^*$ be integral domains with $R^*$ locally affine over $A^*$, but $R^*$ not finitely generated over $A^*$ (e.g. we could take the example of Eakin and Silver mentioned above, $A^* = \mathbb{Z}$ and $R^* = \mathbb{Z}[1/(X/p_i)_{i=1}^\infty]$). Let $K$ be the quotient field of $R^*$ and let $M$ be the maximal ideal of the formal

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2 If $R$ is locally a polynomial ring over $A$, then it is sufficient to have these properties for $D$ a polynomial ring over $A$. Thus, for example, as Eakin and Silver show in [1], if $A$ is a Krull domain, then $R$ locally polynomial over $A$ and contained in an affine ring over $A$ imply $R$ is affine over $A$. This result easily generalizes to the case where $A$ has a locally finite representation, $A = \cap_{\beta} A_{r_{\beta}}$, with the $A_{r_{\beta}}$ rank one valuation rings.
power series ring \( K[[Y]] \). Let \( A = A^* + M \) and \( R = R^* + M \). Then \( M \) is a common prime ideal of \( A \) and \( R \) and \( M \) compares with any other prime ideal of \( A \) or \( R \) (see for example [2, p. 560]). It is now easily verified that \( R \) is locally affine over \( A \) and \( R \supseteq A[1/Y] \), so \( R \) is contained in an affine ring over \( A \). But if \( R \) were finitely generated over \( A \), then \( R^* = R/M \) would be finitely generated over \( A^* = A/M \) which it is not.

REFERENCES


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