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LOCALLY AFFINE RING EXTENSIONS OF A NOETHERIAN DOMAIN

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ABSTRACT. If $A \subset R$ are integral domains with A noetherian, it is shown that if R is contained in an affine ring over A and if for each maximal ideal P of A with $S=A\backslash P$, R_S is an affine ring over A_P, then R itself is affine over A.

If $A \subset R$ are (commutative) integral domains and if for each prime ideal P of A with $S=A\backslash P$, R_S is a finitely generated ring extension of A_P, then we say that R is *locally affine* over A. If for each prime P of A, R_S is a polynomial ring over A_P, then R is said to be *locally a polynomial ring* over A. Even for $A=Z$, the ring of integers, it can happen that R is locally affine over A (and even locally polynomial over A), but yet R is not finitely generated over A. Eakin and Silver in **[1]** give the following example. Let $\{p_i\}_{i=1}^{\infty}$ be the set of prime integers in Z and let X be an indeterminate. Then $R=Z[\{X/p_i\}_{i=1}^{\infty}]$ is locally a polynomial ring in one variable over Z, but R is not finitely generated over Z. The question thus naturally arises of what additional conditions on R, locally affine over A, will imply that R is finitely generated over A. Eakin and Silver **[1]** consider the question of whether R locally a polynomial ring over A and contained in an affine ring over A imply R is finitely generated over A. They prove this to be the case when A is a Krull domain, when A is a pseudo-geometric domain, or when A is a 1-dimensional noetherian domain, and raise the question of whether in general R locally polynomial over a noetherian domain A and contained in an affine ring over A imply R is an affine ring over A. That the answer is yes is a consequence of the following.

THEOREM. *Let $A \subset R$ be integral domains with A noetherian and R locally affine over A. If R is contained in an affine ring over R, then R itself is affine over A.*

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In proving this, our main theorem, we will make use of some properties of the representation of a noetherian integral domain D in the form $D=\bigcap_\alpha \{D_{P_\alpha} | P_\alpha$ is an associated prime of a principal ideal of $D\}$. In general, if $\{V_\alpha\}$ is a collection of integral domains with quotient field K, then $\{V_\alpha\}$ is said to have *finite character* if each nonzero element of K is a unit in all but finitely many of the V_α. If $D=\bigcap_\alpha V_\alpha$ also has quotient field K and $\{V_\alpha\}$ has finite character, then the representation $D=\bigcap_\alpha V_\alpha$ is said to be *locally finite* [3, p. 76]. Note that $D=\bigcap_\alpha V_\alpha$ is locally finite if and only if each nonzero element of D is a unit in all but finitely many of the V_α. For an arbitrary integral domain D, it is well known that if $\{P_\alpha\}$ is a collection of prime ideals of D such that each associated prime of a principal ideal of D is contained in some P_α, then $D=\bigcap_\alpha D_{P_\alpha}$ (see [4, p. 118] or [3, p. 34]). When D is noetherian one has, moreover, the following.

LEMMA 1. *If D is a noetherian domain, then the representation $D=\bigcap_\alpha \{D_{P_\alpha} | P_\alpha$ is an associated prime of a principal ideal of $D\}$ is locally finite.*

PROOF. Since D is noetherian, if P is an associated prime of a principal ideal of D, then P is an associated prime of any nonzero $y \in P$. Thus, nonzero elements of D are contained in only finitely many of the P_α and the lemma follows.

LEMMA 2. *Let D be a noetherian domain, let $\{P_\alpha\}$ be the set of associated primes of principal ideals of D, and let S be a multiplicative system in D. Then $D_S=\bigcap_\alpha \{D_{P_\alpha} | S\cap P_\alpha=\varnothing\}$.*

PROOF. Each associated prime of a principal ideal in D_S is of the form $P_\alpha D_S$ where $P_\alpha \cap S=\varnothing$. Since for $P_\alpha \cap S=\varnothing$, $D_{P_\alpha}=(D_S)_{P_\alpha D_S}$, the result follows.

We note the following immediate consequence of Lemma 2.

LEMMA 3. *Let $D\subset R$ be integral domains with D noetherian and $R\subset D[1/f]$ for some nonzero $f\in D$. Let $\{P_\alpha\}$ be the set of associated primes of principal ideals in D and let $P_1, \cdots, P_n$ be the P_α which contain f. If s is a nonzero element of D such that $R\not\subset D_{P_i}$ implies $s\in P_i$, then $R\subset D[1/s]$.*

We shall also make use of the following standard lemma.

LEMMA 4. *Let A be a commutative ring with identity and let R be an A-algebra. If for each maximal ideal P of A there exists $s\in A\backslash P$ such that $R[1/s]=R\otimes_A A[1/s]$ is a finitely generated $A[1/s]$-algebra, then R is a finitely generated A-algebra.*

PROOF. Let $\{P_\beta\}$ be the set of maximal ideals of A and let $s_\beta \in A\backslash P_\beta$ be such that $R[1/s_\beta]$ is a finitely generated $A[1/s_\beta]$-algebra. Then $(\{s_\beta\})=A$, so there exist $s_1, \cdots, s_n \in \{s_\beta\}$ such that $(s_1, \cdots, s_n)=A$. If N_i is a finite subset of R such that $R[1/s_i]=A[1/s_i, N_i]$, then $N=N_1\cup\cdots\cup N_n$ is such that $R=A[N]$.

PROOF OF THE THEOREM. Let P be a maximal ideal of A, let $S=A\backslash P$, and let $x_1, \cdots, x_n \in R$ be such that $A_S[x_1, \cdots, x_n]=R_S$. By Lemma 4, it will suffice to show for some $s \in S$ that $R[1/s]=A[1/s, x_1, \cdots, x_n]$. Let $D=A[x_1, \cdots, x_n]$. Then $D\subset R$ and R is contained in the quotient field of D. By assumption, R is contained in an affine ring over A, so we have $R\subset D[\xi_1, \cdots, \xi_m]$. By taking a residue class ring of $D[\xi_1, \cdots, \xi_m]$ modulo a prime ideal lying over (0) in R, we may assume that $D[\xi_1, \cdots, \xi_m]$ is an integral domain. Applying the normalization lemma of [**4**, p. 45], we may reduce modulo a suitable prime ideal to the case where the ξ_i are algebraic over D. Finally, writing the ξ_i with denominators in D and using the fact that R is contained in the quotient field of D, we get a nonzero $f\in D$ such that $R\subset D[1/f]$. (A reduction to this case is also given by Eakin and Silver [**1**].) Since A is noetherian, $D=A[x_1, \cdots, x_n]$ is noetherian. Let $\{P_\alpha\}$ be the set of associated primes of principal ideals in D. Note that if $f\notin P_\alpha$, then $D[1/f]\subset D_{P_\alpha}$, so $R\subset D_{P_\alpha}$. Let $P_1, \cdots, P_m$ be the P_α which contain f. If $R\not\subset D_{P_i}$, then $P_i\cap A\not\subset P$; for $R\subset A_S[x_1, \cdots, x_n]$ and $P_i\cap A\subset P$ implies $A_S[x_1, \cdots, x_n]\subset D_{P_i}$. Hence we can choose $s\in A\backslash P$ such that $R\not\subset D_{P_i}$ implies $s\in P_i$. By Lemma 3, $R\subset D[1/s]= A[1/s, x_1, \cdots, x_n]$, which completes the proof of the Theorem.

REMARK. It would be interesting to have more general conditions on A and R in order that R locally affine over A implies that R is finitely generated over A. In our proof, the fact that A is noetherian was used to insure that if D is a domain finitely generated over A and if $\{P_\alpha\}$ is the set of associated primes of principal ideals of D, then the representation $D=\bigcap_\alpha D_{P_\alpha}$ is locally finite and for any nonzero $s\in D$, $D[1/s]= \bigcap_\alpha \{D_{P_\alpha} | s\notin P_\alpha\}$. Perhaps these properties hold for A in a larger class of integral domains, thus allowing generalization of the above theorem by weakening the noetherian hypothesis on A.[2] It is not true in general, however, for integral domains $A\subset R$ with R locally affine over A and contained in an affine ring over A that necessarily R is finitely generated over A. The following example illustrates this.

EXAMPLE. Let $A^*\subset R^*$ be integral domains with R^* locally affine over A^*, but R^* not finitely generated over A^* (e.g. we could take the example of Eakin and Silver mentioned above, $A^*=Z$ and $R^*=Z[\{X/p_i\}_{i=1}^\infty]$). Let K be the quotient field of R^* and let M be the maximal ideal of the formal

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[2] If R is locally a polynomial ring over A, then it is sufficient to have these properties for D a polynomial ring over A. Thus, for example, as Eakin and Silver show in [**1**], if A is a Krull domain, then R locally polynomial over A and contained in an affine ring over A imply R is affine over A. This result easily generalizes to the case where A has a locally finite representation, $A=\bigcap_\beta A_{P_\beta}$, with the A_{P_β} rank one valuation rings.

power series ring $K[[Y]]$. Let $A=A^*+M$ and $R=R^*+M$. Then M is a common prime ideal of A and R and M compares with any other prime ideal of A or R (see for example [**2**, p. 560]). It is now easily verified that R is locally affine over A and $R\subset A[1/Y]$, so R is contained in an affine ring over A. But if R were finitely generated over A, then $R^*=R/M$ would be finitely generated over $A^*=A/M$ which it is not.

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