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INJECTIVE DIMENSION AND COMPLETENESS

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ABSTRACT. This paper contains the proofs of the two following theorems: (1) Let $\{M_\alpha\}_{\alpha<\gamma}$ be a well-ordered decreasing system of submodules of the module M such that $M=M_0$. If M is strongly complete and strongly Hausdorff then

$$\text{inj dim } M \leqq \sup_{\alpha<\gamma} \text{inj dim } M_\alpha/M_{\alpha+1}.$$

(2) Let R be a commutative ring having nonzero minimal idempotent ideals $\{S_\alpha\}_{\alpha<\gamma}$ and let $S=\coprod_{\alpha<\gamma} S_\alpha$. An R-module is injective if and only if $M=\text{Annih } S\oplus M_0$ where Annih S is injective and M_0 is strongly complete and Hausdorff in the topology introduced by annihilators of the direct sums of S_α.

Introduction and notation. In this paper we shall prove a theorem which is in some sense dual to the well-known lemma of Auslander [**1**]. An application of this result is shown.

All the modules considered in this paper are left unitary. Inj dim M denotes the injective dimension of the module M and $E(M)$ stands for the injective hull of the module M.

Let $\{M_\alpha\}$ be a decreasing system of the submodules of M indexed by ordinals less than γ. We have a uniform topology on M whose basis of neighborhoods of zero consists of $\{M_\alpha\}_{\alpha<\gamma}$. Let λ be a limit ordinal less than γ. M/M_λ is topologized by a system of neighborhoods of zero consisting of M_α/M_λ, $\alpha<\lambda$.

If M is complete and M/M_λ is complete for every limit ordinal $\lambda<\gamma$, we say that M is strongly complete. If M is Hausdorff and M/M_λ is Hausdorff for every limit ordinal $\lambda<\gamma$, we say that M is strongly Hausdorff. Putting it in a different way M is strongly Hausdorff if and only if $\bigcap_{\alpha<\gamma} M_\alpha=0$ and $M_\lambda=\bigcap_{\alpha<\lambda} M_\alpha$ for every limit ordinal $\lambda<\gamma$. Let N be a submodule of M. We say that N is strongly closed in M if N is closed in M and $N/N\cap M_\lambda$ is closed in M/M_λ for every limit ordinal $\lambda<\gamma$.

LEMMA 1. *Let N be a strongly complete submodule—in the induced topology—of a strongly Hausdorff module M. Then N is strongly closed in M.*

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LEMMA 2. *N is strongly closed in M if and only if M/N is strongly Hausdorff in the induced topology.*

Proofs are straightforward.

PROPOSITION. *Let N be a strongly closed submodule of a strongly complete module M. Then M/N is strongly complete and strongly Hausdorff in the induced topology.*

PROOF. It follows from Lemma 2 that M/N is strongly Hausdorff. We have to prove that $M/N/M_\lambda+N/N \simeq M/M_\lambda+N$ is complete for every limit ordinal $\lambda<\gamma$.

There is an exact sequence

$$0 \to N/M_\lambda \cap N \to M/M_\lambda \to M/M_\lambda + N \to 0.$$

M/M_λ is a strongly complete module and $N/M_\lambda \cap N$ is strongly closed in M/M_λ. So it suffices to prove that M/N is complete.

Let $\{m_\alpha+N\}_{\alpha<\gamma}$ be a Cauchy Moore-Smith sequence in M/N. We can assume that this sequence is indexed by ordinals less than γ. Moreover we can assume that this sequence is a neat Cauchy sequence i.e.

(1) $(m_\beta + N) - (m_\alpha + N) \in M_\alpha + N$ for every $\alpha < \gamma$ and $\beta \geqq \alpha$.

We put $x_0=m_0$. Suppose we have constructed a sequence $\{x_\alpha\}_{\alpha<\delta}$ where $x_\alpha \in M$ and $\delta<\gamma$ such that $x_\beta-x_\alpha \in M_\alpha$, $\alpha\leqq\beta<\delta$, and $x_\alpha+N=m_\alpha+N$. We claim that there exists $x_\delta \in M$ such that the sequence $\{x_\alpha\}_{\alpha<\delta+1}$ satisfies the above conditions for $\alpha\leqq\beta<\delta+1$. We distinguish two cases.

1°. $\delta-1$ exists. Then by (1) $m_\delta-x_{\delta-1}=m+n$ where $m \in M_{\delta-1}$ and $n \in N$. We put $x_\delta=m_\delta-n$. $x_\delta-x_\alpha=(x_\delta-x_{\delta-1})+(x_{\delta-1}-x_\alpha) \in M_\alpha$ for every $\alpha<\delta$. So x_δ is well chosen.

2°. δ is a limit ordinal. M is a strongly complete module so there exists $x \in M$ such that $x-x_\alpha \in M_\alpha$ for $\alpha<\delta$. $x-m_\alpha \in M_\alpha+N$ because $x_\alpha+N=m_\alpha+N$. By (1), $m_\delta-m_\alpha \in M_\alpha+N$. It follows that $x-m_\delta \in M_\alpha+N$. We obtain that $x-m_\delta \in M_\delta+N$ because M/N is strongly Hausdorff. $x-m_\delta=m+n$ with $m \in M_\delta$ and $n \in N$. We put $x_\delta=x-m$.

By the inductive argument there exists in M a Cauchy Moore-Smith sequence $\{x_\alpha\}_{\alpha<\gamma}$ such that $x_\alpha+N=m_\alpha+N$ for $\alpha<\gamma$. Let x denote a limit of this sequence. Then $x+N$ is the limit of $\{m_\alpha+N\}_{\alpha<\gamma}$.

THEOREM 1. *Let $\{M_\alpha\}_{\alpha<\gamma}$ be a well-ordered decreasing system of submodules of the module M such that $M=M_0$. If M is strongly complete and strongly Hausdorff then*

$$\text{inj dim } M \leqq \sup_{\alpha<\gamma} \text{inj dim } M_\alpha/M_{\alpha+1}.$$

PROOF. Let $n = \sup_{\alpha<\gamma} \operatorname{inj\,dim} M_\alpha/M_{\alpha+1}$. Suppose $n=0$. Then all the modules $M_\alpha/M_{\alpha+1}$ are injective. For every $\alpha<\gamma$ we define $f_\alpha: M \to M_\alpha/M_{\alpha+1}$ as an extension of the natural homomorphism $M_\alpha \to M_\alpha/M_{\alpha+1}$. All the homomorphisms f_α define a homomorphism $f: M \to \prod_{\alpha<\gamma} M_\alpha/M_{\alpha+1}$. Let $0 \neq m \in M$. M is Hausdorff so there exists $\alpha<\gamma$ such that $m \notin M_\alpha$. Let α be the least ordinal with this property. α is not a limit ordinal because M is strongly Hausdorff. We obtain that $f_{\alpha-1}(m) \neq 0$. So $f(m) \neq 0$. We have proved that f is a monomorphism. Our claim is that f is also an epimorphism.

Let $\{m_\alpha + M_{\alpha+1}\}_{\alpha<\gamma} \in \prod_{\alpha<\gamma} M_\alpha/M_{\alpha+1}$. We put $x_0 = m_0$. Suppose we have a sequence $\{x_\tau\}_{\tau<\delta}$ where $x_\tau \in M$ and $\delta<\gamma$ such that $x_\beta - x_\tau \in M_\tau$ for $\tau \leqq \beta < \delta$ and $f_\alpha(x_\tau) = m_\alpha + M_{\alpha+1}$ for $\alpha \leqq \tau$. We shall find $x_\delta \in M$ such that the sequence $\{x_\tau\}_{\tau<\delta+1}$ satisfies above conditions for $\tau \leqq \beta < \delta+1$. There are two cases.

1°. $\delta-1$ exists. The homomorphism f_δ is an extension of the natural homomorphism $M_\delta \to M_\delta/M_{\delta+1}$. So there is $m \in M_\delta$ such that $f_\delta(m) = (m_\delta + M_{\delta+1}) - f(x_{\delta-1})$. We put $x_\delta = x_{\delta-1} + m$.

2°. δ is a limit ordinal. M is a strongly complete module. So there exists $x \in M$ such that $x - x_\tau \in M_\tau$ for $\tau<\delta$. We put $x_\delta = x + m$ where m is an element of M_δ such that $f_\delta(m) = (m_\delta + M_{\delta+1}) - f_\delta(x)$. It is easy to see that in both cases x_δ is well chosen.

By induction there exists a Cauchy sequence $\{x_\tau\}_{\tau<\gamma}$ in M such that $f_\alpha(x_\tau) = m_\alpha + M_{\alpha+1}$ for $\alpha \leqq \tau < \gamma$. Let x denote a limit of this sequence. Then $f_\alpha(x) = m_\alpha + M_{\alpha+1}$ for $\alpha<\gamma$ as it is easy to show. So $f(x) = (m_\alpha + M_{\alpha+1})_{\alpha<\gamma}$. We have proved that M is injective because it is a module isomorphic to the direct product of injectives.

Let now $\sup_{\alpha<\gamma} \operatorname{inj\,dim} M_\alpha/M_{\alpha+1} = n > 0$. We put $E = \prod_{\alpha<\gamma} E(M_\alpha/M_{\alpha+1})$. Let π_α denote a projection of E onto $E(M_\alpha/M_{\alpha+1})$. We define $E_0 = E$ and $E_\alpha = E_{\alpha-1} \cap \ker \pi_{\alpha-1}$ if $\alpha-1$ exists, $E_\alpha = \bigcap_{\tau<\alpha} E_\tau$ in the opposite case. It is obvious that E is strongly complete and strongly Hausdorff in the topology introduced by E_α, $\alpha<\gamma$. In the same way as above we define $f: M \to E = \prod_{\alpha<\gamma} E(M_\alpha/M_{\alpha+1})$ which is a monomorphism. Topology on M coincides with the topology induced by E because $E_\alpha \cap M = M_\alpha$. By Lemmas 1 and 2 and the Proposition, E/M is strongly complete and strongly Hausdorff in the induced topology. There is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_\alpha & \longrightarrow & E_\alpha & \longrightarrow & E_\alpha + M/M & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M_{\alpha+1} & \longrightarrow & E_{\alpha+1} & \longrightarrow & E_{\alpha+1} + M/M & \longrightarrow & 0 \end{array}$$

$$[E_\alpha + M/M \simeq E_\alpha/M \cap E_\alpha = E_\alpha/M_\alpha].$$

All the vertical homomorphisms are monomorphisms. So we obtain an exact sequence of cokernels

$$0 \to M_\alpha/M_{\alpha+1} \to E_\alpha/E_{\alpha+1} \to E_\alpha + M/E_{\alpha+1} + M \to 0.$$

$E_\alpha/E_{\alpha+1}$ is injective because $E_\alpha/E_{\alpha+1} \simeq E(M_\alpha/M_{\alpha+1})$. We obtain that inj dim $E_\alpha + M/E_{\alpha+1} + M \leqq n-1$ for every $\alpha < \gamma$. By inductive hypothesis inj dim $E/M \leqq n-1$. So inj dim $M \leqq n$ because E is injective.

Let $\{S_\alpha\}_{\alpha<\gamma}$ be a set of idempotent minimal ideals of a commutative ring R indexed by ordinals less than γ—starting with zero. For any R-module M we put $M_\alpha = \text{Annih} \coprod_{\tau<\alpha} S_\tau$ for $\alpha \geqq 0$ ($S_{-1}=0$). M is Hausdorff if and only if Annih $S=0$ where $S = \coprod_{\alpha<\gamma} S_\alpha$.

THEOREM 2. *Let R be a commutative ring having nonzero idempotent minimal ideals. An R-module M is injective if and only if* $M = \text{Annih } S \oplus M_0$ *where* Annih S *is injective and* M_0 *is strongly complete and Hausdorff.*

PROOF. Let M be injective. We claim that Annih S is also injective. It is sufficient to prove that Annih S has no essential extension in M. Let x be an element of M such that $Ix \subset \text{Annih } S$ where I is an essential ideal. Then $Sx = S^2x \subset SIx = 0$—the inclusion follows from the fact that every essential ideal contains the socle—so $x \in \text{Annih } S$. It follows that $M = \text{Annih } S \oplus M_0$ where M_0 is injective and Hausdorff. We shall prove that M_0 is strongly complete. Let $\{m_\alpha\}_{\alpha<\gamma}$ be a neat Cauchy sequence in M_0, i.e. $m_\beta - m_\alpha \in M_\alpha$ for $\beta \geqq \alpha$. Let $r \in S$. Then $r \in \coprod_{\tau<\alpha} S_\tau$ for some $\alpha < \gamma$. We define $f(r) = rm_\alpha$. It is easy to check that f is a well-defined homomorphism from S into M_0. By injectivity of M_0, $f(r) = rm$ for every $r \in S$ and some $m \in M_0$. An element m obtained above is a limit of the sequence $\{m_\alpha\}_{\alpha<\gamma}$. So M_0 is complete. The strong completeness of this module is proved similarly.

To finish the proof of the theorem we must show that strong completeness and Hausdorffness of M implies its injectivity. M is obviously strongly Hausdorff. By Theorem 1 it suffices to prove that $M_\alpha/M_{\alpha+1}$ is injective. It is easy to show that $M_\alpha/M_{\alpha+1}$ is annihilated by a maximal ideal $\mathfrak{m}_\alpha$ such that $\mathfrak{m}_\alpha \oplus S_\alpha = R$. So $M_\alpha/M_{\alpha+1}$ is an injective $R/\mathfrak{m}_\alpha$-module—$R/\mathfrak{m}_\alpha$ is a field. $M_\alpha/M_{\alpha+1}$ is also injective as an R-module because $\mathfrak{m}_\alpha$ is a direct summand of R.

REMARK. Theorem 2 gives us a structure theorem for algebraically compact modules over commutative regular rings with a nonzero socle because in this case notions of algebraic compactness and injectivity coincide. The result obtained in this way is in some sense similar to the Kaplansky structure theorem for algebraically compact groups.

COROLLARY. *Let R satisfy the conditions of Theorem 2 and let M be a directed union of a countable family of injective submodules* $\{M_i\}_{i<\omega}$ *and* *such that* Annih $\coprod_{i<\omega} S_i=0$ *where* ω *is the first countable ordinal. Then* M *is injective if and only if there exist finite ordinals* j *and* k *such that* Annih $\coprod_{i<j} S_i \subset M_k$.

Proof. Suppose M is injective. Then M is complete and is a union of a countable family of closed subsets M_i, $i<\omega$. M_i are complete subsets of M which is Hausdorff. By the Baire theorem one of these subsets has a nonempty interior. So Annih $\coprod_{i<j} S_i \subset M_k$ for some j and k. To prove the theorem in the other direction it suffices to remark that our assumption implies that M/M_k is discrete because M_k is a direct summand of M. So M is isomorphic to the direct sum of injective modules M_k and M/M_k and therefore is injective.

Reference

1. M. Auslander, *On the dimension of modules and algebras.* III. *Global dimension*, Nagoya Math. J. **9** (1955), 67–77. MR **17**, 579.

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