INTEGRABLY PARALLELIZABLE MANIFOLDS
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Abstract. A smooth manifold $M^n$ is called integrably parallelizable if there exists an atlas for the smooth structure on $M^n$ such that all differentials in overlap between charts are equal to the identity map of the model for $M^n$. We show that the class of connected, integrably parallelizable, $n$-dimensional smooth manifolds consists precisely of the open parallelizable manifolds and manifolds diffeomorphic to the $n$-torus.

1. Introduction. In this note $M^n$ is an $n$-dimensional, paracompact smooth manifold without boundary, and $G$ is an arbitrary subgroup of the general linear group $\text{Gl}(n, R)$ on $R^n$.

Definition. $M^n$ is called $G$-reducible if the structural group of the tangent bundle for $M^n$ can be reduced from $\text{Gl}(n, R)$ to $G$. $M^n$ is called integrably $G$-reducible if there exists an atlas $\{(U_i, \theta_i)\}$ for the smooth structure on $M^n$ such that the differential in overlap between charts $(\theta_i \circ \theta_j^{-1})_x$ belongs to $G$ for all $x \in \theta_j(U_i \cap U_j) \subseteq \mathbb{R}^n$ and all $i, j$ in the index set for the atlas.

It is clear that an integrably $G$-reducible manifold is $G$-reducible and therefore the following problem naturally arises.

Problem. Classify for a given $G$ those $G$-reducible manifolds which are integrably $G$-reducible.

Let us illustrate this problem with two examples.

Example 1. Let $G = \text{O}(n)$ be the orthogonal group. Any manifold $M^n$ is $\text{O}(n)$-reducible, since it admits a Riemannian metric. On the other hand it is easy to see that $M^n$ is integrably $\text{O}(n)$-reducible if and only if it admits a flat Riemannian metric. (A flat Riemannian manifold is locally isometric to $\mathbb{R}^n$.)

Example 2. Suppose $n = 2k$ and let $\text{Gl}(k, \mathbb{C})$ be the general linear group on $\mathbb{C}^k$ considered as a subgroup of $\text{Gl}(n, R)$ under the usual identification of $\mathbb{C}^k$ with $\mathbb{R}^n$. Then $M^n$ is $\text{Gl}(k, \mathbb{C})$-reducible if and only if it admits an almost complex structure and integrably $\text{Gl}(k, \mathbb{C})$-reducible if and only if it admits a complex structure. The classification of the manifolds admitting a complex structure among those admitting an almost complex structure is far from being complete.
For open manifolds Haefliger has recently given a nice reformulation of the problem using the work on foliations of Phillips and Gromov. See e.g. Haefliger [3, Example 2]. Notice here that an integrably G-reducible manifold in our sense in Haefliger’s terminology is a manifold which admits a G-structure. For closed manifolds very little seems to be known.

The purpose of this note is to give the complete solution to the problem in the case where G = G0 is the identity subgroup of GL(n, R). It is easy to see that a manifold M" is G0-reducible if and only if it is parallelizable. Therefore we will call an integrably G0-reducible manifold integrably parallelizable. With this notation we have

**Theorem.** Let M" be connected and parallelizable. Then M" is integrably parallelizable in precisely the following cases:

(i) M" is open;
(ii) M" is diffeomorphic to the n-dimensional torus T^n = S^1 \times \cdots \times S^1.

This theorem answers a question of J. Eells, who asked for a determination of the finite dimensional integrably parallelizable smooth manifolds after the discovery that any smooth separable Hilbert manifold is integrably parallelizable in a very strong sense (see the remark in §3).

2. **Proof of the theorem.** For the proof of the theorem we need two well-known results which we state as lemmas. First a lemma of Frobenius type (see e.g. Hicks [4, p. 128]).

**Lemma 1.** Let M" be a parallelizable smooth manifold parallelized by the smooth vector fields, X1, \ldots, Xn. Suppose that these vector fields commute, i.e. all Lie brackets [X_i, X_j] = 0. Then each point x \in M" has a coordinate neighbourhood (U, \theta) such that the restrictions of the vector fields X_1, \ldots, X_n to U coincide with the coordinate vector fields on U.

Recall now that the rank of a smooth manifold M" is the maximal number of linearly independent, commuting smooth vector fields which can be defined on the manifold. Then we have

**Lemma 2.** A compact, connected smooth manifold M" has rank n if and only if it is diffeomorphic to the n-torus T^n = S^1 \times \cdots \times S^1.

**Proof.** This result was originally obtained by Willmore [8, Theorem 2]. We offer here an alternative proof. Assume that M" has rank n. There is then an action of R^n (considered with its standard abelian Lie group structure) on M" such that all the orbits for the action are immersed submanifolds; see e.g. Rosenberg [7]. For an arbitrary point x \in M" consider now the isotropy group R^n_x for such an action at x and its quotient group G_x = R^n/R^n_x in R^n. The orbit map o_x : R^n \to M" at x induces then a map
\[ \delta_x : G_x \to M^n. \]

Since \( R^n \) is a closed discrete subgroup of \( R^n \), \( G_x \) can be given the structure of an abelian Lie group such that the canonical projection \( R^n \to G_x \) is a Lie group homomorphism. It is easy to prove that \( \delta_x \) is surjective and then it follows immediately that \( \delta_x \) is a diffeomorphism. But then \( G_x \) is a compact, connected abelian Lie group and hence diffeomorphic to \( T^n \). Thus \( M^n \) is also diffeomorphic to \( T^n \). Since conversely \( T^n \) clearly has rank \( n \) the proof is finished.

**Proof of the theorem.** (i) Suppose first that \( M^n \) is open. By a result of Hirsch [5, Theorem 4.7] there exists then an immersion \( F : M^n \to R^n \). Locally this immersion is a diffeomorphism and we can therefore define an atlas \( \{(U_i, \theta_i)\} \) on \( M^n \), such that \( \theta_i = F|_{U_i} \). But then it is clear that all differentials in overlap between charts \( (\theta_i \circ \theta_j^{-1})_* = 1_{R^n} \), the identity on \( R^n \). Hence \( M^n \) is integrably parallelizable.

(ii) Suppose now that \( M^n \) is compact. If we can show that \( M^n \) is integrably parallelizable if and only if it has rank \( n \), then Lemma 2 will finish the proof of the theorem. Suppose therefore first that \( M^n \) has rank \( n \) and choose \( n \) linearly independent, commuting smooth vector fields \( X_1, \ldots, X_n \) on \( M^n \). By Lemma 1 we can define an atlas \( \{(U_i, \theta_i)\} \) on \( M^n \) such that the coordinate vector fields on \( U_i \) coincide with the restrictions of the vector fields \( X_1, \ldots, X_n \) to \( U_i \). But then it is easy to see that all differentials \( (\theta_i \circ \theta_j^{-1})_* = 1_{R^n} \) and hence \( M^n \) is integrably parallelizable. Suppose next that \( M^n \) is integrably parallelizable and let \( \{(U_i, \theta_i)\} \) be an atlas on \( M^n \) with all differentials \( (\theta_i \circ \theta_j^{-1})_* = 1_{R^n} \). It is then easy to see that there exist \( n \) well-defined smooth vector fields \( X_1, \ldots, X_n \) on \( M^n \), whose restrictions to \( U_i \) coincide with the coordinate vector fields on \( U_i \) for any chart \( (U_i, \theta_i) \) in the atlas. Since coordinate vector fields commute it is clear that \( X_1, \ldots, X_n \) are \( n \) linearly independent, commuting smooth vector fields on \( M^n \). Hence \( M^n \) has rank \( n \). As already observed this finishes the proof.

3. **Examples and a remark.** In this section we give first some examples of integrably parallelizable manifolds.

**Example 3.** Every open Lie group or punctured compact, connected Lie group is integrably parallelizable.

**Example 4.** Every orientable 3-manifold is parallelizable. Hence the class of connected integrably parallelizable 3-manifolds consists of all open connected orientable 3-manifolds plus diffeomorphic images of the 3-torus.

**Example 5.** Let \( V_{n,k} \) be the Stiefel manifold of orthonormal \( k \)-frames in \( R^n \). By a theorem of Borel and Hirzebruch [1] this is always a \( \pi \)-manifold (stably trivial tangent bundle). Since an open \( \pi \)-manifold is parallelizable (an open manifold \( M^n \) has a complex of dimension \( n-1 \) as
deformation retract) it follows that all punctured Stiefel manifolds are integrably parallelizable.

Example 6. Let $M^n$ be any compact, connected $\pi$-manifold. Then $M^n \times S^1$ is parallelizable but not integrably parallelizable unless $M^n$ is diffeomorphic to $T^n$. $M^n \times \mathbb{R}^1$ is also parallelizable but as an open manifold now even integrably parallelizable.

We finish with a remark concerning infinite dimensional manifolds.

Remark. Let $X$ be an infinite dimensional separable smooth manifold modelled on the separable Hilbert space $E$. By a theorem of Kuiper [6] the general linear group $\text{Gl}(E)$ on $E$ is contractible. This implies that $X$ is parallelizable. By the recent theorem of Eells and Elworthy [2] $X$ is diffeomorphic to an open subset of $E$. The parallelization of $X$ can therefore be realized by a single coordinate chart. This implies of course that $X$ is integrably parallelizable in a very strong sense. In finite dimensions it would have been too much to ask for realization of a parallelization by a single coordinate chart. The punctured 2-torus e.g. is integrably parallelizable in our sense but is not diffeomorphic to an open subset of $\mathbb{R}^2$.

References


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