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INTEGRABLY PARALLELIZABLE MANIFOLDS

VAGN LUNDSGAARD HANSEN

ABSTRACT. A smooth manifold M^n is called integrably parallelizable if there exists an atlas for the smooth structure on M^n such that all differentials in overlap between charts are equal to the identity map of the model for M^n. We show that the class of connected, integrably parallelizable, n-dimensional smooth manifolds consists precisely of the open parallelizable manifolds and manifolds diffeomorphic to the n-torus.

1. **Introduction.** In this note M^n is an n-dimensional, paracompact smooth manifold without boundary, and G is an arbitrary subgroup of the general linear group $\mathrm{Gl}(n, \boldsymbol{R})$ on $\boldsymbol{R}^n$.

DEFINITION. M^n is called *G-reducible* if the structural group of the tangent bundle for M^n can be reduced from $\mathrm{Gl}(n, \boldsymbol{R})$ to G. M^n is called *integrably G-reducible* if there exists an atlas $\{(U_i, \theta_i)\}$ for the smooth structure on M^n such that the differential in overlap between charts $(\theta_i \circ \theta_j^{-1})_{*x}$ belongs to G for all $x \in \theta_j(U_i \cap U_j) \subseteq R^n$ and all i, j in the index set for the atlas.

It is clear that an integrably G-reducible manifold is G-reducible and therefore the following problem naturally arises.

PROBLEM. Classify for a given G those G-reducible manifolds which are integrably G-reducible.

Let us illustrate this problem with two examples.

EXAMPLE 1. Let $G=O(n)$ be the orthogonal group. Any manifold M^n is $O(n)$-reducible, since it admits a Riemannian metric. On the other hand it is easy to see that M^n is integrably $O(n)$-reducible if and only if it admits a flat Riemannian metric. (A flat Riemannian manifold is locally isometric to $\boldsymbol{R}^n$.)

EXAMPLE 2. Suppose $n=2k$ and let $\mathrm{Gl}(k, \boldsymbol{C})$ be the general linear group on $\boldsymbol{C}^k$ considered as a subgroup of $\mathrm{Gl}(n, \boldsymbol{R})$ under the usual identification of $\boldsymbol{C}^k$ with $\boldsymbol{R}^n$. Then M^n is $\mathrm{Gl}(k, \boldsymbol{C})$-reducible if and only if it admits an almost complex structure and integrably $\mathrm{Gl}(k, \boldsymbol{C})$-reducible if and only if it admits a complex structure. The classification of the manifolds admitting a complex structure among those admitting an almost complex structure is far from being complete.

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For open manifolds Haefliger has recently given a nice reformulation of the problem using the work on foliations of Phillips and Gromov. See e.g. Haefliger [**3**, Example 2]. Notice here that an integrably G-reducible manifold in our sense in Haefliger's terminology is a manifold which admits a G-structure. For closed manifolds very little seems to be known.

The purpose of this note is to give the complete solution to the problem in the case where $G=G_0$ is the identity subgroup of $\mathrm{Gl}(n, \boldsymbol{R})$. It is easy to see that a manifold M^n is G_0-reducible if and only if it is parallelizable. Therefore we will call an integrably G_0-reducible manifold *integrably parallelizable*. With this notation we have

THEOREM. *Let M^n be connected and parallelizable. Then M^n is integrably parallelizable in precisely the following cases:*

(i) *M^n is open;*

(ii) *M^n is diffeomorphic to the n-dimensional torus $T^n=S^1\times\cdots\times S^1$.*

This theorem answers a question of J. Eells, who asked for a determination of the finite dimensional integrably parallelizable smooth manifolds after the discovery that any smooth separable Hilbert manifold is integrably parallelizable in a very strong sense (see the remark in §3).

2. **Proof of the theorem.** For the proof of the theorem we need two well-known results which we state as lemmas. First a lemma of Frobenius type (see e.g. Hicks [**4**, p. 128]).

LEMMA 1. *Let M^n be a parallelizable smooth manifold parallelized by the smooth vector fields, $X_1, \cdots, X_n$. Suppose that these vector fields commute, i.e. all Lie brackets $[X_i, X_j]=0$. Then each point $x\in M^n$ has a coordinate neighbourhood (U, θ) such that the restrictions of the vector fields $X_1, \cdots, X_n$ to U coincide with the coordinate vector fields on U.*

Recall now that the rank of a smooth manifold M^n is the maximal number of linearly independent, commuting smooth vector fields which can be defined on the manifold. Then we have

LEMMA 2. *A compact, connected smooth manifold M^n has rank n if and only if it is diffeomorphic to the n-torus $T^n=S^1\times\cdots\times S^1$.*

PROOF. This result was originally obtained by Willmore [**8**, Theorem 2]. We offer here an alternative proof. Assume that M^n has rank n. There is then an action of $\boldsymbol{R}^n$ (considered with its standard abelian Lie group structure) on M^n such that all the orbits for the action are immersed submanifolds; see e.g. Rosenberg [7]. For an arbitrary point $x\in M^n$ consider now the isotropy group $\boldsymbol{R}^n_x$ for such an action at x and its quotient group $G_x=\boldsymbol{R}^n/\boldsymbol{R}^n_x$ in $\boldsymbol{R}^n$. The orbit map $o_x:\boldsymbol{R}^n\rightarrow M^n$ at x induces then a map

$\bar{o}_x: G_x \to M^n$. Since $\boldsymbol{R}_x^n$ is a closed discrete subgroup of $\boldsymbol{R}^n$, G_x can be given the structure of an abelian Lie group such that the canonical projection $\boldsymbol{R}^n \to G_x$ is a Lie group homomorphism. It is easy to prove that o_x is surjective and then it follows immediately that $\bar{o}_x$ is a diffeomorphism. But then G_x is a compact, connected abelian Lie group and hence diffeomorphic to T^n. Thus M^n is also diffeomorphic to T^n. Since conversely T^n clearly has rank n the proof is finished.

Proof of the theorem. (i) Suppose first that M^n is open. By a result of Hirsch [**5**, Theorem 4.7] there exists then an immersion $F: M^n \to \boldsymbol{R}^n$. Locally this immersion is a diffeomorphism and we can therefore define an atlas $\{(U_i, \theta_i)\}$ on M^n, such that $\theta_i = F|U_i$. But then it is clear that all differentials in overlap between charts $(\theta_i \circ \theta_j^{-1})_{*x} = 1_{\boldsymbol{R}^n}$, the identity on $\boldsymbol{R}^n$. Hence M^n is integrably parallelizable.

(ii) Suppose now that M^n is compact. If we can show that M^n is integrably parallelizable if and only if it has rank n, then Lemma 2 will finish the proof of the theorem. Suppose therefore first that M^n has rank n and choose n linearly independent, commuting smooth vector fields $X_1, \cdots, X_n$ on M^n. By Lemma 1 we can define an atlas $\{(U_i, \theta_i)\}$ on M^n such that the coordinate vector fields on U_i coincide with the restrictions of the vector fields $X_1, \cdots, X_n$ to U_i. But then it is easy to see that all differentials $(\theta_i \circ \theta_j^{-1})_{*x} = 1_{\boldsymbol{R}^n}$ and hence M^n is integrably parallelizable. Suppose next that M^n is integrably parallelizable and let $\{(U_i, \theta_i)\}$ be an atlas on M^n with all differentials $(\theta_i \circ \theta_j^{-1})_{*x}^{-1} = 1_{\boldsymbol{R}^n}$. It is then easy to see that there exist n well-defined smooth vector fields $X_1, \cdots, X_n$ on M^n, whose restrictions to U_i coincide with the coordinate vector fields on U_i for any chart (U_i, θ_i) in the atlas. Since coordinate vector fields commute it is clear that $X_1, \cdots, X_n$ are n linearly independent, commuting smooth vector fields on M^n. Hence M^n has rank n. As already observed this finishes the proof.

3. **Examples and a remark.**

In this section we give first some examples of integrably parallelizable manifolds.

Example 3. Every open Lie group or punctured compact, connected Lie group is integrably parallelizable.

Example 4. Every orientable 3-manifold is parallelizable. Hence the class of connected integrably parallelizable 3-manifolds consists of all open connected orientable 3-manifolds plus diffeomorphic images of the 3-torus.

Example 5. Let $V_{n,k}$ be the Stiefel manifold of orthonormal k-frames in $\boldsymbol{R}^n$. By a theorem of Borel and Hirzebruch [**1**] this is always a π-manifold (stably trivial tangent bundle). Since an open π-manifold is parallelizable (an open manifold M^n has a complex of dimension $n-1$ as deformation retract) it follows that all punctured Stiefel manifolds are integrably parallelizable.

EXAMPLE 6. Let M^n be any compact, connected π-manifold. Then $M^n \times S^1$ is parallelizable but not integrably parallelizable unless M^n is diffeomorphic to T^n. $M^n \times \boldsymbol{R}^1$ is also parallelizable but as an open manifold now even integrably parallelizable.

We finish with a remark concerning infinite dimensional manifolds.

REMARK. Let X be an infinite dimensional separable smooth manifold modelled on the separable Hilbert space $\boldsymbol{E}$. By a theorem of Kuiper [6] the general linear group $\mathrm{Gl}(\boldsymbol{E})$ on $\boldsymbol{E}$ is contractible. This implies that X is parallelizable. By the recent theorem of Eells and Elworthy [2] X is diffeomorphic to an open subset of $\boldsymbol{E}$. The parallelization of X can therefore be realized by a single coordinate chart. This implies of course that X is integrably parallelizable in a very strong sense. In finite dimensions it would have been too much to ask for realization of a parallelization by a single coordinate chart. The punctured 2-torus e.g. is integrably parallelizable in our sense but is not diffeomorphic to an open subset of $\boldsymbol{R}^2$.

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MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, ENGLAND

MATHEMATICS INSTITUTE, UNIVERSITY OF AARHUS, AARHUS, DENMARK

Current address: Mathematics Institute, University of Copenhagen, Copenhagen, Denmark