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ON EMBEDDING ANNULI IN M^3

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ABSTRACT. Let M be a 3-manifold. In this note we give a condition when a "nontrivial" proper map of an annulus into M^3 can be replaced by a "nontrivial" proper embedding of an annulus in M^3.

Let λ_1 and λ_2 be disjoint simple loops embedded in the boundary of an orientable 3-manifold M. We suppose that λ_1 and λ_2 are freely homotopic in M and that λ_1 is not nullhomotopic in M. Then it is a consequence of a well-known theorem in [3] that $\lambda_1 \cup \lambda_2$ is the boundary of an annulus embedded in M. Unfortunately we can obtain no new information from this theorem when λ_1 and λ_2 bound an annulus in the boundary of M.

The following theorem allows us to construct a "nontrivial annulus" in a few special cases when the free homotopy of λ_1 and λ_2 cannot be deformed to $\mathrm{bd}(M)$. This theorem seems to be closely related to a problem of R. H. Fox discussed in [**1**].

The proof of Theorem 1 is basically an application of the proof of the sphere theorem in [**2**] and [**4**]. One might hope that one could use a tower of two sheeted coverings and obtain a more general result. The difficulty here is analogous to the one encountered when trying to prove the sphere theorem via a tower of two sheeted coverings, i.e. one obtains an embedded surface at the top of the tower, but the projection of the surface may not be nontrivial in the original sense.

Throughout the remainder of this paper all spaces will be simplicial complexes and all maps will be piecewise linear.

Let l be a simple loop embedded in a 3-manifold M and based at a point x. Let $\langle [l] \rangle$ be the subgroup of $\pi_1(M, x)$ generated by $[l]$. Let $N(l)$ be the normal closure of $\langle [l] \rangle$ in $\pi_1(M, x)$ and $C(l)$ the centralizer of $\langle [l] \rangle$ in $\pi_1(M, x)$.

THEOREM 1. *Let M be a compact, orientable 3-manifold such that $\pi_2(M) = 0$. Let F be a surface, other than a torus, which is a boundary component of M. Let l be a simple loop embedded in F and not nullhomotopic in M. Then*

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if $C(l)-N(l)$ is nonempty, there is an annulus A properly embedded in M which cannot be deformed to the boundary of M while keeping the boundary of A fixed.

PROOF. Let $a \in C(l)-N(l)$. Let l_a be a loop in M representing a. We divide the boundary of a disk D into four segments and map these segments into M as is illustrated in Figure 1. Since the loop $l_a l l_a^{-1} l^{-1}$ is null-homotopic in M, we can extend the map defined above to all of D. We identify the segments of the boundary of D, which are mapped to l_a, to obtain an annulus A_1 and a map $f:(A_1, \mathrm{bd}(A_1)) \to (M, l)$.

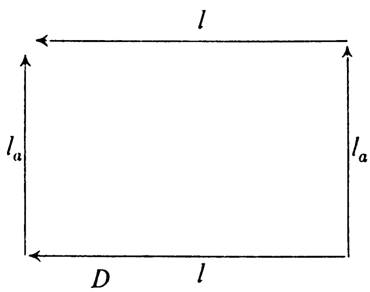

FIGURE 1

Now we claim $f:(A_1, \mathrm{bd}(A_1)) \to (M, l)$ is not homotopic rel $\mathrm{bd}(A_1)$ to a map into the boundary of M. Otherwise l_a could be deformed to lie in F and $[l]$ would commute with a in $\pi_1(F)$. Since F is not a torus, a would be in $\langle [l] \rangle$. This is a contradiction.

Thus it is easily shown that there is a proper map f of an annulus A into M where $f|\mathrm{bd}(A)$ is a homeomorphism and f cannot be deformed to lie in $\mathrm{bd}(M)$ under a deformation constant on $\mathrm{bd}(A)$. We attach a 3-cell C to M along l to obtain a 3-manifold M_1. We attach 2-cells D_1 and D_2 to A to obtain a 2-sphere S. We extend $f:A \to M_1$ to a map $f_1:S \to M_1$ which is a homeomorphism on $D_1 \cup D_2$. We may assume that f_1 carries the interior of $D_1 \cup D_2$ into C. We can put the map f_1 into "general position with respect to itself" via a deformation constant outside of $M \subset M_1$ so that $f_1(S)$ is a canonical singular 2-sphere (see [2]).

We claim that $f_1:S \to M_1$ is essential. Let (M_1^*, p) be the universal cover of M_1. Since $a \notin N(l)$, l_a does not lift to a closed path in M_1^*. Thus if we consider any map $f_1^*:S \to M_1^*$ such that $pf_1^* = f_1$, $f_1^*(D_1)$ and $f_1^*(D_2)$ lie in different components of $p^{-1}(C)$. If f_1 is inessential, $[f_1^*(S)]$ would represent the trivial element of $H_2(M_1^*, p^{-1}(M)) \simeq H_2(p^{-1}(C), p^{-1}(C \cap M))$, a direct sum of infinite cyclic groups. Since the same element picks up generators from two direct summands, this is a contradiction.

It follows from the proof of the sphere theorem [2] and [4] that there is an essential embedding g of the 2-sphere S in M_1 and that we may assume $g(S)\cap C$ is a collection of zero, one, or two disks nicely embedded in C. Since $\pi_2(M)=0$ and $g(S)$ is essential, $g(S)\cap C$ contains at least one disk. Since l is not nullhomotopic in M, we see that there cannot be exactly one disk in $g(S)\cap C$. Thus $g(S)\cap M$ is an annulus. If the annulus $g(S)\cap M$ can be deformed to the boundary of M under a deformation constant on the boundary of $g(S)\cap M$, it can be deformed to lie in $C\cap M$ under a deformation constant on the boundary of $g(S)\cap M$ since F is not a torus. Since $g(S)$ is essential in M_1 and C is a cell this is impossible. This completes the proof of our theorem.

Full details of a proof of the more general "annulus theorem" reported by F. Waldhausen would be most interesting.

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