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A SPLITTING RING OF GLOBAL DIMENSION TWO[1]

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ABSTRACT. In this paper an example is given of a ring with left global dimension 2 having the property that the singular submodule of any R-module A is a direct summand of A. Although the example given is quite specific, the methods can be used to construct a fairly large class of these rings.

In this paper, all rings are assumed to be associative with an identity element, and all modules will be unitary left modules.

An R-module A is said to split if the singular submodule, $Z({}_RA)$, is a direct summand of A. R is called a splitting ring if every R-module splits (see [1], [3], and [7]). In [3] Cateforis and Sandomierski have shown that every commutative splitting ring has left global dimension $\leqq 1$. M. L. Teply [7] has shown that if the commutative hypothesis is dropped, then every splitting ring must have left global dimension $\leqq 2$. Several splitting rings of left global dimension 1 were known, but no such rings of left global dimension 2 have been found; thus the question arises, which is the best bound, 1 or 2? In this paper it is shown that 2 is the best possible bound for the left global dimension of a splitting ring.

R is said to have the finitely generated splitting property (FGSP) if every finitely generated R-module splits. Cateforis and Sandomierski [3] have shown that every commutative ring with FGSP must be semihereditary. A trivial consequence of the example constructed in this paper is that a (noncommutative) ring with FGSP need not be left or right semihereditary.

For an R-module A, let $\mathrm{soc}(A)$ denote the socle of A. If R is a ring of matrices, we define e_{ij} to be the matrix with the identity element of the appropriate coordinate ring in the ith row and jth column and zeros elsewhere.

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Let S be a ring, and let M be an essential maximal left ideal of S which is also a two-sided ideal. Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \middle| \, d \in S;\, a, b, c, e, f \in S/M \right\},$$

and let

$$\Lambda = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, b \in S;\, a, c \in S/M \right\}.$$

We wish to choose S such that R will become the desired splitting ring of left global dimension two. But first we point out a few basic properties of R:

LEMMA 1. (a) *R is a left (right) Noetherian ring if and only if S is a left (right) Noetherian ring.*

(b) *If $Z({}_S S)=0$, then $Z({}_R R)=0$.*

(c) *If S has no nontrivial idempotent elements, then* l.gl.dim $R \geqq 2$.

PROOF. (a) The "if" part is an immediate consequence of the fact that ${}_S R$ is finitely generated. The "only if" follows from the existence of a (ring) homomorphism of R onto S given by

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \to d.$$

(b) If $Z({}_S S)=0$, it is straightforward to check that annihilators of elements in the essential left ideal $Re_{11} \oplus Re_{22} \oplus Re_{13}$ are not essential in R. Hence $Z({}_R R)=0$.

(c) If l.gl.dim $R \leqq 1$, then

$$(0 : e_{23}) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix} \middle| \, c \in M;\, a, b, d, e \in S/M \right\}$$

is generated by an idempotent element

$$\begin{pmatrix} u & 0 & v \\ 0 & w & x \\ 0 & 0 & y \end{pmatrix}.$$

This forces $w^2 = w \in M$, which contradicts the hypothesis that S contains no nontrivial idempotent elements.

Now let C be a left and right principal ideal domain with the following properties:

(a) C is a simple ring, which is not a division ring;

(b) every simple C-module is injective;

(c) there exists (up to isomorphism) only one simple C-module.

Examples of such rings have been provided by Cozzens [**4**]. Let M be a maximal left ideal of C, and let $I=I_C(M)=\{x \in C \mid mx \in M \text{ for all } m \in M\}$ be the idealizer of M in C (see [**6**]). By [**6**, Theorem 4.3], I is a hereditary Noetherian integral domain with nontrivial, two-sided ideal $M=M^2$. By [**6**, Theorem 1.3 and Corollary 2.4], I has only two simple modules S_1 and S_2 (up to isomorphism); S_1 is a faithful injective simple module, $S_2 \simeq I/M$, and $E(S_2)/S_2 \simeq S_1$ (where $E(S_2)$ denotes the injective envelope of S_2). By [**8**, Theorem 4], every nonzero singular I-module has a nonzero socle.

We now give a sequence of lemmas designed to show that $I=S$ makes R the desired splitting ring.

LEMMA 2. *I is a splitting ring.*

PROOF. It is sufficient to show that $\mathrm{Ext}_I^1(F, T)=0$ for any nonsingular I-module F and any singular I-module T.

As noted above, any singular I-module has nonzero socle; so $\mathrm{soc}(E(T))=\mathrm{soc}\, T$ is essential in $E(T)$. Write $\mathrm{soc}(T)=X\oplus Y$, where every simple submodule of X is isomorphic to S_1 and every simple submodule of Y is isomorphic to S_2. Since S_1 is injective and I is Noetherian, then X is injective and $E(T)\simeq X\oplus E(Y)$. Moreover, either $E(Y)=0$ or else $E(Y)=E(\oplus\sum_{\beta\in\mathscr{B}} S_2^{(\beta)})\simeq\oplus\sum_{\beta\in\mathscr{B}} E(S_2^{(\beta)})$ for some index set $\mathscr{B}$, where $S_2^{(\beta)}\simeq S_2$ for all $\beta \in \mathscr{B}$. Hence $\mathscr{B}\neq\varnothing$ implies

$$E(T)/\mathrm{soc}(T) \simeq \oplus \sum_{\beta\in\mathscr{B}} (E(S_2)/S_2) \simeq \oplus \sum_{\beta\in\mathscr{B}} S_1^{(\beta)}$$

with $S_1^{(\beta)}\simeq S_1$ for all $\beta \in \mathscr{B}$. Since I is hereditary and Noetherian, it follows that $E(T)/\mathrm{soc}(T)$ is injective. But $T/\mathrm{soc}(T)$ can be embedded (as a summand) in $E(T)/\mathrm{soc}(T)$; whence $T/\mathrm{soc}(T)$ is also injective. Hence we have the exact sequence

$$\mathrm{Ext}_I^1(F, \mathrm{soc}(T)) \to \mathrm{Ext}_I^1(F, T) \to \mathrm{Ext}_I^1(F, T/\mathrm{soc}(T)) = 0.$$

Thus it is sufficient to show that $\mathrm{Ext}_I^1(F, \mathrm{soc}(T))=0$.

Since X is injective,

$$\mathrm{Ext}_I^1(F, \mathrm{soc}(T)) \simeq \mathrm{Ext}_I^1(F, X \oplus Y)$$
$$\simeq \mathrm{Ext}_I^1(F, X) \oplus \mathrm{Ext}_I^1(F, Y) \simeq \mathrm{Ext}_I^1(F, Y).$$

From [5, Theorem 5.2], it follows that F is a flat I-module. Since I/M is a division ring, then $\mathrm{Ext}_I^1(F, \mathrm{soc}(T)) \cong \mathrm{Ext}_I^1(F, Y) \cong \mathrm{Ext}_{I/M}^1(I/M \otimes_I F, Y) = 0$ by [2, VI, Proposition 4.1.3]. Hence I is a splitting ring.

Let

$$N = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \middle| \, d \in M;\, a, b, c, e, f \in I/M \right\},$$

and let U be the simple R-module R/N.

LEMMA 3. $\mathrm{Ext}_\Lambda^n(R, U) = 0$ *for* $n \geqq 1$.

PROOF. Since Λ is a hereditary ring, $\mathrm{Ext}_\Lambda^n(R, U) = 0$ for $n \geqq 2$. As a left Λ-module, $R = \oplus \sum \{\Lambda e_{ij} \mid 1 \leqq i \leqq j \leqq 3\}$. Since Λe_{11}, Λe_{12}, Λe_{13}, Λe_{22}, Λe_{33} are Λ-projective, then

$$\mathrm{Ext}_\Lambda^1(R, U) \cong \oplus \sum \{\mathrm{Ext}_\Lambda^1(\Lambda e_{ij}, U) \mid 1 \leqq i \leqq j \leqq 3\} \cong \mathrm{Ext}_\Lambda^1(\Lambda e_{23}, U).$$

Let $0 \to U \to X \xrightarrow{\varphi} \Lambda e_{23} \to 0$ be an exact sequence of Λ-modules. Since $e_{11}\varphi(X) = e_{11}U = 0$, $e_{11}X = 0$. Similarly $e_{33}X = 0$. Thus X has identical Λ- and I-module structures, namely, $X \cong U \oplus \Lambda e_{23}$ (since $E(S_2)/S_2 \cong S_1$ and $S_2 \cong U \cong \Lambda e_{23}$ as I-modules). Therefore $\mathrm{Ext}_\Lambda^1(\Lambda e_{23}, U) = 0$.

LEMMA 4. $\mathrm{Hom}_\Lambda(R, U) \cong U \oplus U$.

PROOF. Λe_{11}, Λe_{12}, Λe_{13}, and Λe_{33} are nonsingular simple Λ-modules, and U is a singular simple Λ-module. Therefore

$$\mathrm{Hom}_\Lambda(\Lambda e_{11} \oplus \Lambda e_{12} \oplus \Lambda e_{13} \oplus \Lambda e_{33}, U) = 0.$$

Since $\Lambda e_{22} \cong I$ and $\Lambda e_{23} \cong U$ each have identical Λ- and I-module structures and since ${}_IU$ is annihilated by M, then

$$\begin{aligned} \mathrm{Hom}_\Lambda(R, U) &\cong \mathrm{Hom}_\Lambda(\Lambda e_{22} \oplus \Lambda e_{23}, U) \cong \mathrm{Hom}_I(I \oplus U, U) \\ &\cong \mathrm{Hom}_I(I, U) \oplus \mathrm{Hom}_I(U, U) \cong U \oplus U. \end{aligned}$$

LEMMA 5. $\mathrm{Ext}_R^n(A, U \oplus U) \cong \mathrm{Ext}_\Lambda^n(A, U)$ *for all R-modules A and for all* $n \geqq 1$.

PROOF. By Lemma 3 and [2, VI, Proposition 4.1.4],

$$\mathrm{Ext}_R^n(A, \mathrm{Hom}_\Lambda(R, U)) \cong \mathrm{Ext}_\Lambda^n(A, U)$$

for all R-modules A and for all $n \geqq 1$. Thus the conclusion follows from Lemma 4.

LEMMA 6. $\mathrm{Ext}_\Lambda^1(A, U) = 0$ *for all nonsingular R-modules A.*

Proof. Since $\Lambda \cong I/M \oplus I/M \oplus I$, it follows from Lemma 2 that Λ is a splitting ring. Therefore, $A = Z({}_\Lambda A) \oplus F$ for any nonsingular R-module A, and $\operatorname{Ext}^1_\Lambda(F, U) = 0$. So it suffices to show that $\operatorname{Ext}^1_\Lambda(Z({}_\Lambda A), U) = 0$.

If $0 \neq t \in Z({}_\Lambda A)$, then the Λ-annihilator of t is

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \; b \in M;\, a, c \in I/M \right\}.$$

For let K denote the R-annihilator of t, and let L be the set of elements of I which appear in the second row and second column of some element of K. Since $0 \neq t \in Z({}_\Lambda A)$, L is a nontrivial left ideal of I. If $m \in M$, then

$$\left(K : \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \supseteq \left\{ \begin{pmatrix} x & y & z \\ 0 & u & v \\ 0 & 0 & w \end{pmatrix} \middle| \; u \in (ML : m);\, x, y, z, u, v \in I/M \right\}$$

is essential in R. Since R/K is a nonsingular R-module, this forces

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} \in K.$$

It is now easy to verify that the Λ-annihilator has the desired form.

Thus $Z({}_\Lambda A)$ is 0 or a direct sum of simple R-modules isomorphic to U. As it was noted in the proof of Lemma 3, $\operatorname{Ext}^1_\Lambda(U, U) = 0$. Consequently $\operatorname{Ext}^1_\Lambda(Z({}_\Lambda A), U) = 0$.

Lemma 7. $\operatorname{Ext}^1_R(A, B) = 0$, *where A is any nonsingular R-module and B is any R-module isomorphic to a direct sum of copies of U.*

Proof. Let $B = \oplus \sum_{\alpha \in \mathscr{A}} U_\alpha$, where $U_\alpha \cong U$. Consider the exact sequence

$$(*) \qquad 0 \to \oplus \sum_{\alpha \in \mathscr{A}} U_\alpha \xrightarrow{f} \prod_{\alpha \in \mathscr{A}} U_\alpha \to \prod_{\alpha \in \mathscr{A}} U_\alpha \Big/ \oplus \sum_{\alpha \in \mathscr{A}} U_\alpha \to 0.$$

By Lemmas 5 and 6, $\operatorname{Ext}^1_R(A, \prod_{\alpha \in \mathscr{A}} U_\alpha) \cong \prod_{\alpha \in \mathscr{A}} \operatorname{Ext}^1_R(A, U_\alpha) = 0$. Thus $(*)$ induces an exact sequence

$$\operatorname{Hom}_R\left(A, \prod_{\alpha \in \mathscr{A}} U_\alpha\right) \xrightarrow{f_*} \operatorname{Hom}_R\left(A, \prod_{\alpha \in \mathscr{A}} U_\alpha \Big/ \oplus \sum_{\alpha \in \mathscr{A}} U_\alpha\right)$$
$$\longrightarrow \operatorname{Ext}^1_R\left(A, \oplus \sum_{\alpha \in \mathscr{A}} U_\alpha\right) \to 0.$$

Since $NU = 0$, then $(*)$ is also an exact sequence of R/N-modules. Since

R/N is a division ring, $(*)$ splits as R/N-modules and as R-modules. Therefore f_* is an epimorphism, and hence $\operatorname{Ext}_R^1(A, \oplus \sum_{\alpha\in\mathscr{A}} U_\alpha)=\operatorname{Ext}_R^1(A, B)=0$ by exactness.

LEMMA 8. *Every simple singular R-module $V\not\cong U$ is R-injective.*

PROOF. If V is a simple singular R-module not isomorphic to U, then $V\cong R/T$, where T is of one of the following types of maximal left ideals of R:

(a) $$T=\left\{\begin{pmatrix} a & b & c\\ 0 & d & e\\ 0 & 0 & f\end{pmatrix}\middle|\ d\in L;\ a,b,c,e,f\in I/M\right\},$$

where L is a maximal left ideal of I distinct from M;

(b) $$T=\left\{\begin{pmatrix} a & b & c\\ 0 & d & e\\ 0 & 0 & 0\end{pmatrix}\middle|\ d\in I;\ a,b,c,e\in I/M\right\}.$$

To establish that V is injective, it is sufficient to show that for any essential left ideal K of R and for any diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & R\\ & & {\scriptstyle g}\downarrow & & \\ & & V & & \end{array}$$

there is an $h: R\rightarrow V$ such that $hf=g$. The left ideal K can be any one of the following types, where H denotes a nonzero left ideal of I.

(A) $$K=\left\{\begin{pmatrix} x & y & z\\ 0 & u & 0\\ 0 & 0 & 0\end{pmatrix}\middle|\ u\in H;\ x,y,z\in I/M\right\}.$$

(B) $$K=\left\{\begin{pmatrix} x & y & z\\ 0 & u & v\\ 0 & 0 & 0\end{pmatrix}\middle|\ u\in H;\ x,y,z,v\in I/M\right\}.$$

(C) $$K=\left\{\begin{pmatrix} x & y & z\\ 0 & u & v\\ 0 & 0 & w\end{pmatrix}\middle|\ u\in H;\ x,y,z,v,w\in I/M\right\}.$$

(D) $$K=\left\{\begin{pmatrix} x & y & z\\ 0 & u & v(u)\\ 0 & 0 & 0\end{pmatrix}\middle|\ \begin{array}{l} u\in H;\ x,y,z,v(u)\in I/M;\\ \qquad\qquad \text{if } u=0, \text{ then } v(u)=0\end{array}\right\}.$$

If K is of type (D), the situation can be reduced to type (C) by the following argument. Let

$$P = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} \middle| \, a, b, c \in I/M \right\}.$$

Then

$$\begin{pmatrix} x & y & z \\ 0 & u & s \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y & z \\ 0 & u & v(u) \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s - v(u) \\ 0 & 0 & 0 \end{pmatrix} \in K + P.$$

Therefore $K+P$ is of type (C). Since $Z(_R V)=V$ and since $\mathrm{soc}(K)$ is nonsingular by Lemma 1(b), then $K \cap P \subseteq \mathrm{soc}(K) \subseteq \ker g$. Hence g can be extended to $K+P$ by setting $g(P)=0$. Therefore the situation is reduced to type (C).

Let T be of type (a). If K is of type (A), (B) or (C), then g can be extended to all of R by using the fact that V is an injective I-module.

If T is of type (b) and K is of type (A) or (B), then $e_{33}K=0$ and $e_{33}V\neq 0$. Thus $g(K)=0$, and the zero map extends g.

Finally, if T is of type (b) and K is of type (C), then g can be extended to $h: R \to V$ by setting $h(x)=g(e_{33}x)$ for each $x \in R$.

THEOREM 9. *R is a splitting ring with* $\mathrm{l.gl.dim}\, R=2$.

PROOF. To show R is a splitting ring, it is sufficient to show that $\mathrm{Ext}^1_R(F, T)=0$ for any nonsingular R-module F and for any singular R-module T. By Lemma 1(a), Lemma 8, and the fact that $N^2=N$, $T/\mathrm{soc}(T)$ is an injective, semisimple R-module. As in the proof of Lemma 2, it can be assumed that $\mathrm{soc}(T)$ is a direct sum of copies of U. By Lemma 7, $\mathrm{Ext}^1_R(F, \mathrm{soc}(T))=0$. From the exact sequence

$$0 = \mathrm{Ext}^1_R(F, \mathrm{soc}(T)) \to \mathrm{Ext}^1_R(F, T) \to \mathrm{Ext}^1_R(F, T/\mathrm{soc}(T)) = 0,$$

it follows that $\mathrm{Ext}^1_R(F, T)=0$. Therefore R is a splitting ring.

Since any splitting ring R satisfies $Z(_R R)=0$ [3], then Lemma 1(c) and [**7**, Theorem 2.2] yield $\mathrm{l.gl.dim}\, R=2$.

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REFERENCES

1. J. S. Alin and S. E. Dickson, *Goldie's torsion theory and its derived functor*, Pacific J. Math. **24** (1968), 195–293. MR **37** #2834.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956. MR **17**, 1040.

3. V. C. Cateforis and F. L. Sandomierski, *The singular submodule splits off*, J. Algebra **10** (1968), 149–165. MR **39** #2805.

4. J. H. Cozzens, *Homological properties of the ring of differential polynomials*, Bull. Amer. Math. Soc. **76** (1970), 75–79. MR **41** #3531.

5. L. Levy, *Torsion-free and divisible modules over non-integral domains*, Canad. J. Math. **15** (1963), 132–151. MR **26** #155.

6. J. C. Robson, *Idealizers and hereditary Noetherian prime rings*, J. Algebra (to appear).

7. M. L. Teply, *Homological dimension and splitting torsion theories*, Pacific J. Math. **34** (1970), 193–205. MR **43** #291.

8. D. B. Webber, *Ideals and modules of simple Noetherian hereditary rings*, J. Algebra **16** (1970), 239–242. MR **42** #305.

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