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A NOTE ON JANKO'S SIMPLE GROUP OF ORDER 175,560

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ABSTRACT. Janko's simple group J of order 175,560 is characterized among simple groups by the weak closure W of the involution in its centralizer. Among arbitrary finite groups, the theorem asserts that the normal closure of W is J.

1. **Introduction.** The existence of a simple group J of order 175,560 was established by Professor Z. Janko in [6]. Indeed, Janko proved that the group J was characterized by the presence of an involution t such that

(a) t lies in the center of a 2-Sylow subgroup of G,

(b) $C_G(t) \simeq \langle t \rangle \times A_5$,

(c) t is not central in G.

In this paper we prove the following:

THEOREM. *Let t be an involution in a group G. Assume* (i) *t lies in the center of a 2-Sylow subgroup of G and* (ii) *the weak closure of t in its centralizer in G has the form $\langle t \rangle \times A_5$. Then $\langle t^G \rangle$ is isomorphic to Janko's simple group of order* 175,560.

This "weak closure" version of a centralizer-characterization for J plays a key role in the study (being carried out by Professor M. Herzog and the author) of groups whose proper central 2-Sylow intersections are cyclic or generalized quaternion groups.

2. **Proof of the theorem.** The proof proceeds by a series of short steps. Let G be a minimal counterexample.

(1) *t lies in no proper normal subgroup of G of 2-power index.*

Assume $t \in N \trianglelefteq G$ where G/N is a 2-group. Since by (i) $|t^G|$ is odd, N transitively permutes the elements of t^G so

$$t^G = t^N.$$

Since all conjugates of t lie in N, the weak closure of t in $C_G(t)$ relative to G is also the weak closure of t in $C_N(t)$ relative to N. Thus (i) and (ii)

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hold with N in place of G. If $|N|<|G|$, we may imply induction on $|N|$ to obtain

$$\langle t^G \rangle = \langle t^N \rangle \simeq J,$$

where J denotes Janko's group. Since this presents us with the conclusion of the lemma, we may assume $|N|=|G|$. This allows us to assume (1).

(2) *Set* $W_t = \langle g^{-1}tg \mid g^{-1}tg \in C_G(t), g \in G \rangle$, *the weak closure of* t *in* $C_G(t)$. *Let* T *be a fixed* 2-*Sylow subgroup of* W_t. *Then, by* (ii), T *is elementary of order* 8. T *is weakly closed in any* 2-*Sylow subgroup which contains* T. *Also* $N(T)$ *controls fusion in* T.

Let S be a 2-Sylow subgroup of $C(t)$. Then $S \cap W_t$ is a 2-Sylow subgroup of W_t and so without loss of generality we may assume that $S \cap W_t = T$. Since W_t is generated by conjugates of t there exist a conjugate of t, say t^g in $W_t - \langle t \rangle$. Then, by conjugating by elements in W_t we may assume $t^g \in T$. Then, by conjugating by elements in $N_{W_t}(T)$ we see that conjugates of t generate T, whence

$$S \cap W_t = T = \langle t^G \cap S \rangle. \tag{2.1}$$

Thus T is the weak closure of t in S relative to G. This forces T to be weakly closed in S—i.e., T is the unique conjugate of T lying in S. It follows that any 2-Sylow subgroup of G contains only one conjugate of T. Thus T is weakly closed in any 2-Sylow subgroup containing it. It follows from Sylow's theorem that T is weakly closed in any 2-subgroup of G containing T.

If a and a^g both lie in T, then T and $T^{g^{-1}}$ lie in $C(a)$. Then there exists an element $c \in C(a)$ such that $T^{g^{-1}c}$ and T lie in a common 2-Sylow subgroup of $C(a)$. From the last line of the previous paragraph $T^{g^{-1}c} = T$. Thus $g^{-1}c \in N(T)$ and so $c^{-1}g \in N(T)$. Then $a^{c^{-1}g} = a^g$ and so the fusion $a \to a^g$ can be achieved in $N(T)$.

All assertions in (2) have been proved.

(3) $T^{\#}$ *is fused in* G (*and in* $N(T)$).

Since S is a 2-Sylow subgroup of $N(T)$ and fusion in T occurs in $N(T)$ we have $|t^G \cap T|$ is odd. Now in $W_t \cap N_G(T)$, there exists an element of order 3 normalizing T, and stabilizing and acting fixed point free on a subgroup of T complementing $\langle t \rangle$. Indeed, $W_t \cap N_G(T)$ acts on T with orbits of lengths 1, 3 and 3, and t belongs to the orbit of length one. Since T is the weak closure of t in S, $t^G \cap S = t^G \cap T$ is a union of the $W_t \cap N_G(T)$-orbits mentioned above, has an odd number of elements and contains more than one element. It follows that $t^G \cap T = T^{\#}$ and (3) is proved.

(4) T *lies in the center of every* 2-*Sylow subgroup containing it.*

Since $[W_t, W_t] \simeq A_5$ is a normal subgroup of $C(t)$, we see that $C(t) \cap N(T)$ contains $[W_t, W_t] \cap N(T) \simeq A_4$ as a normal subgroup. Thus

$N(T)/C(T)$ is isomorphic to a subgroup of $SL(3, 2)$ which is (a) transitive on the seven elements of $T^{\#}$ and (b) in which the subgroup $(N(T)\cap C(t))/C(T)$ fixing one of the elements of T, lies in the normalizer of

$$([W_t, W_t] \cap N(T))C(T) \mid C(T) \simeq Z_3,$$

corresponding to a 3-Sylow subgroup of $SL(3, 2)$. It follows that $[N(T):C(T)]=21$ or 42. Because the 7-Sylow normalizers of $SL(3, 2)$ are maximal in $SL(3, 2)$ we see that $N(T)/C(T)$ is the nonabelian group of order 21. Thus a 2-Sylow subgroup of G lying in $N(T)$ lies in $C(T)$. Since T is weakly closed in any 2-Sylow subgroup of G containing it, it follows that T lies in the center of every 2-Sylow subgroup containing it.

(5) *Let $\mathscr{C}=\mathscr{C}(t^G)$ be the graph whose vertices are t^G, and whose arcs are commuting pairs of involutions in t^G. Let $\mathscr{C}_1$ denote the connected component of $\mathscr{C}$ containing t. Every element of odd order in $C(T)$ fixes $\mathscr{C}_1$ pointwise.*

Let u denote an element of odd order in $C(T)$. Since A_5 admits no automorphism of odd order fixing pointwise one of its 2-Sylow subgroups, we see that for each $S \in T^{\#}$, u normalizes $W_s=\mathrm{Vccl}_G(s, C_G(s))\simeq\langle s\rangle \times A_5$ and hence centralizes each W_s. What this means is that if $T^g \cap T$ is nontrivial then y also centralizes T^g. Since every commuting pair of involutions in t^G lies in a conjugate of T, we have that y centralizes every involution belonging to $\mathscr{C}_1$.

$$(6) \qquad O_{2'}(G) = 1.$$

It is easy to see that hypotheses (i) and (ii) inherit to $G/O_{2'}(G)$. By induction, if $O_{2'}(G)\neq 1$, $\langle t^G\rangle O_{2'}(G)/O_{2'}(G)\simeq J$. If $U=[C(t)\cap O_{2'}(G), T]$, then U is a subgroup of W_t having odd orders and is normalized by T, a 2-Sylow subgroup of W_t. From the isomorphism type of W_t, $U=1$. Then (using (3)), $C(t_1)\cap O_{2'}(G)=C(T)\cap O_{2'}(G)$ for all involutions t_1 in $T^{\#}$. Since T is noncyclic, $\langle C(t_1)\cap O_{2'}(G) | t_1 \in T^{\#}\rangle=O_{2'}(G)$ and so the previous sentence implies $O_{2'}(G)\leqq C(T)$. Thus $O_{2'}(G)$ is centralized by $\langle t^g\rangle$. It follows that $\langle t^G\rangle$ is a perfect central extension of J by a central group Z of odd order. But since every odd Sylow subgroup of J is cyclic and has a fixed-point-free element normalizing it, J has no multipliers of odd order. Thus $\langle t^G\rangle$ splits over Z. But since it is generated by involutions and $|Z|$ is odd, it follows that $\langle t^G\rangle\simeq J$, our desired conclusion. Thus we may assume $O_2(G)=1$.

(7) *Any two elements of 2-power order in $N(T)$ which are conjugate in G are conjugate in $N(T)$.*

Suppose x and $g^{-1}xg=x^g$ are two elements of 2-power order in $N(T)$. By (4), x and x^g lie in $C(T)$. Then T and $T^{g^{-1}}$ lie in $C(x)$ and so $T^{g^{-1}c}$ and T lie in a common 2-Sylow subgroup of $C(x)$ for an appropriate choice of c. Since by (2), T is weakly closed in any 2-group containing it,

$T^{g^{-1}c}=T$ so $g^{-1}c \in N(T)$. Then $c^{-1}g \in N(T)$ and $x^g=x^{c^{-1}g}$ is conjugate to x by an element in $N(T)$.

(8) *$\mathscr{C}$ is not connected.*

Suppose by way of contradiction that $\mathscr{C}$ is connected, so that $\mathscr{C}=\mathscr{C}_1$. Let K be the centralizer in G of t^G, so

$$K = \bigcap C(s), \quad s \text{ ranging over } t^G. \tag{2.2}$$

Our first objective will be to show that K is trivial.

Set $K_0=K\cap\langle t^G\rangle$. Then K_0 coincides with center of $\langle t^G\rangle$ and has 2-power order since $Z(\langle t^G\rangle)$ is necessarily a 2-group by (6).

Suppose g is an element in G such that conjugation by g leaves the coset tK_0 fixed. Then $t^g=tk$ where $k \in K_0$. Since $[t, k]=1$ and t^g is an involution, either $k=1$ or k is an involution. In any event $k \in T$ since by (2) and (3), T is the weak closure of $\langle t\rangle$ in S where S is any 2-Sylow subgroup of G containing T. Assume $k\neq 1$. Again by (3), $T^{\#}$ is fused so $k \in t^G$. Then k is a member of t^G commuting with all other members of t^G. Since G acts transitively on t^G, this means that all members of t^G are mutually commuting. Then $\langle t^G\rangle$ is elementary abelian. But then, on the other hand,

$$\langle t^G\rangle = \langle g^{-1}tg \mid g \in G,\ g^{-1}tg \in C_G(t)\rangle = W_t \simeq Z_2 \times A_5,$$

a contradiction. Thus $k=1$. Indeed, we have proved two things:

$$t^G \cap K_0 = \varnothing, \tag{2.3}$$

$$C_{G/K_0}(tK_0) = C_G(t)/K_0. \tag{2.4}$$

Because of (2.3) and (2.4), hypotheses (i) and (ii) hold for G/K_0. Thus if $K_0\neq 1$, induction on G/K_0 yields the fact that $\langle t^G\rangle$ is a perfect central extension of J by K_0. But then J has a perfect central extension by $K_0/K_{00}\simeq Z_2$. But in that case, TK_0/K_0 is a 2-Sylow subgroup of $\langle t^G\rangle/K_0\simeq J$, and TK_0/K_0 is elementary of order 8 and admits an automorphism of order 7. It follows that every coset of $K_0/K_{00}\simeq Z_2$ in TK_0/K_{00} consists entirely of involutions. Thus TK_0/K_{00} is elementary and so the 2-Sylow subgroups of $\langle t^G\rangle/K_{00}$ split over K_0/K_{00}. By a well-known theorem of Gaschütz [2] this implies that $\langle t^G\rangle/K_{00}$ splits over K_0/K_{00}, contrary to the fact that $\langle t^G\rangle/K_{00}$ is a perfect group. Thus we must assume

$$K_0=1. \tag{2.5}$$

Thus

(2.6) the centralizer of tK/K in G/K is covered by $C(t)$.

Also

$$t^G \cap K \text{ is empty} \tag{2.7}$$

is an immediate consequence of (2.3). Now by (2.6) and (2.7), hypotheses (i) and (ii) hold for G/K. Then if $K\neq 1$, induction yields that $\langle t^G\rangle K/K\simeq J$ and so $\langle t^G\rangle$ is a central extension of J by $K\cap\langle t^G\rangle$. But now (2.5) implies $\langle t^G\rangle\simeq J$, our conclusion. Thus we must assume

$$(2.8) \qquad K = 1.$$

Now by (5), (2.8) implies that $C(T)$ is a 2-Sylow subgroup of G. Then $N(T)=SB$ where B is metacyclic of order 21. Let B_1 denote a 3-Sylow subgroup of B. Then B_1 is a 3-Sylow subgroup of $W_s\cap N(T)\simeq Z_2\times A_4$ for some involution s in $T^{\#}$. Then since $S\leqq N(W_s)$, $[S, B_1]\leqq BT$. Since B is generated by its 3-Sylow subgroups $[S, B]\leqq BT$, also. Since B normalizes $C(T)=S$, we have $[S, B]\leqq BT\cap S=T$. Since B and S have coprime orders, $C_S(B)T=S$. Now B acts without fixed points on T so $C_S(B)\cap BT\leqq C_S(B)\cap(BT\cap S)=C_S(B)\cap T=\langle 1\rangle$. Since T is central in S, $C_S(B)=C_S(BT)$. Thus we have

$$(2.9) \qquad C_S(BT)\times BT = N(T).$$

Suppose $C_S(BT)\neq 1$. Then $N(T)$ has a nontrivial 2-factor whose associated kernel contains t. Since $N(T)$ controls its fusion of 2-elements by (7), the focal subgroup of S is proper in S and contains t. It follows from the focal subgroup theorem [**5**] (see Theorems 3.4 and 3.5 of [**4**]) that G contains a proper normal subgroup N of 2-power index and N contains t. But this contradicts step (1). Thus $C_S(BT)=1$ and we now have

$$(2.10) \qquad N(T) = BT, \quad T \text{ is a 2-Sylow subgroup of } G.$$

A Frattini argument now yields $C(t)=(N(T)\cap C(t))W_t$. But $N(T)\cap C(t)$ has the form TB_1 where B_1 is an appropriate 3-Sylow subgroup of B, and $TB_1\leqq W_t$. Thus

$$(2.11) \qquad C(t) = W_t \simeq Z_2\times A_5.$$

That $\langle t^G\rangle\simeq J$ follows from (2.11) and (i) is a theorem of Janko [**6**]. Thus, on the assumption that $\mathscr{C}$ is connected we reach our desired conclusion. Thus we may assume that $\mathscr{C}$ is not connected, which is (8).

(9) *Let H_1 be the stabilizer in G of the set $\mathscr{C}_1$. Then H_1 is a proper subgroup of G and has the form $H_1\simeq Y\times J$, where $t^G\cap H_1\subseteq J$. The centralizer* (*in G*) *of any involution in $Y\cup J$ lies in H_1.*

Clearly for any $t_1\in\mathscr{C}_1$, $C(t_1)\leqq H_1$. By (4), H_1 satisfies hypotheses (i) and (ii). Since $\mathscr{C}$ is not connected by (8) and G is transitive on the vertices of $\mathscr{C}$ (since these are t^G), it follows that H_1 is a proper subgroup of G. Then induction on $|H_1|$ yields $\langle t^{H_1}\rangle\simeq J$. Since $t^g\in\mathscr{C}_1$ implies $g\in H_1$ (since the connected components $\mathscr{C}_1,\cdots$ form a system of imprimitivity on $\mathscr{C}$), we have

$$(2.12) \qquad t^G\cap H_1 = t^{H_1}.$$

Now $\langle t^{H_1}\rangle$ is a normal subgroup of H_1 isomorphic to J. Since J is a complete and simple group,

$$H_1 = C_{H_1}(\langle t^{H_1}\rangle) \times \langle t^{H_1}\rangle \simeq Y \times J$$

where $Y \simeq C_{H_1}(\langle t^{H_1}\rangle)$.

Since an involution in J belongs to $\mathscr{C}_1$, we have already seen that the centralizer of any involution in J lies in H_1.

Now let x be an involution in Y. Suppose, for some $g \in G$, $t^g = tu$ where $u \in Y$. By (2.12), $u \in t^G \cap H_1 = t^{H_1} \subseteq J$. Then $u \in Y \cap J = \langle 1\rangle$. Since $t^G \cap Y = \varnothing$ this means that for any element $w \in Y$, $C_G(w)$ contains W_t and $C_G(w)/\langle w\rangle$ satisfies hypotheses (i) and (ii). In particular, induction yields $\langle t^{C(x)}\rangle\langle x\rangle/\langle x\rangle \simeq J$. But since $\langle t^{H_1}\rangle$ lies in $C(x)$ and is isomorphic to J, it follows that $\langle t^{C(x)}\rangle \leqq H_1$. Thus $C(x)$ stabilizes $t^{C(x)} = t^{H_1} = \mathscr{C}_1$, whence $C(x) \leqq H_1$.

(10) *Any involution* $y \in H_1 - (Y \cup J)$ *satisfies* $C(y) \leqq H_1$.

Let y be an involution in $H_1 - (Y \cup J)$ so $y = y_1 y_2$ where y_1 is an involution in Y and y_2 is an involution in J. Then $\mathscr{C}_1 \cap C(y)$ consists of 31 involutions distributed in W_{y_2}-orbits of lengths 1, 15 and 15 with representatives $t_1 = y_2$, t_2 and t_3, respectively. We see that $\langle \mathscr{C}_1 \cap C(y)\rangle \simeq Z_2 \times A_5$ and without loss of generality we may assume t_3 lies in the A_5-part—i.e. $t_3 \in [W_{t_1}, W_{t_1}]$. If t_3 were conjugate to t_1 or t_2 in $C(y)$, then this fusion must occur in H_1 since the connected components of $\mathscr{C}$ form a system of imprimitivity. But since $C(y) \cap H_1$ has the form $C_y(y_1) \times C_J(t_1)$ this fusion cannot take place. We thus see that

$$\langle t_3^{C(y)\cap H_1}\rangle \simeq A_5. \tag{2.13}$$

Now $C_G(t_3) \leqq H_1$ as we have already established. Thus,

$$C(y) \cap C(t_3) \cap t_3^{C(y)} \subseteq (C(y) \cap H_1) \cap t_3^{C(y)}.$$

Again, since the connected components of $\mathscr{C}$ are a system of imprimitivity

$$C(y) \cap H_1 \cap t_3^{C(y)} = t_3^{C(y)\cap H_1}.$$

Thus,

$$C(y) \cap C(t_3) \cap t_3^{C(y)} \subseteq \langle t_3^{C(y)\cap H_1}\rangle \cap C(t_3)$$

and, by (2.13), is elementary abelian since it corresponds to the centralizer of an involution in A_5.

Thus the weak closure of t_3 in its centralizer in $C(y)$ is an abelian group T_1 of order 4. In addition by (4), t_3 lies in the center of a 2-Sylow subgroup of $\mathscr{C}(y)$. It follows from the corollary to the fusion-theorem in [7] (see also [**3**, p. 62]) that $\langle t_3^{C(y)}\rangle$ is a direct product of Bender groups and a central product of central extensions of groups $U(3, q_i)$ and a 2-nilpotent group having an elementary 2-Sylow subgroup (the centers entering into the central product having odd order). But T_1 must be a 2-Sylow center of this normal product. Also, $\langle t_3^{C(y)\cap H_1}\rangle \simeq A_5$ is a subgroup of $\langle t_3^{C(y)}\rangle$. It follows that either

$$\langle t_3^{C(y)}\rangle \simeq A_5 \tag{2.14}$$

or

$$\langle t_3^{C(y)}\rangle \simeq U(3, 4)^* \tag{2.15}$$

where the asterisk indicates a perfect central extension of $U(3, 4)$ by a group of odd order (necessarily a 3-group). But in that case, $\langle t_3^{C(y)\cap H_1}\rangle$ corresponds to a subgroup of $U(3, 4)^*$ isomorphic to A_5, no two conjugates of which share a common involution (since the involutions of each lie in distinct connected components of $\mathscr{C}$). Since this is clearly impossible, (2.14) holds. Then $t_3^{C(y)} \subseteq \mathscr{C}_1$ and is stabilized by $C(y)$. It follows that $C(y) \leqq H_1$.

(11) *H_1 is strongly embedded in G.*

This follows at once since H_1 contains a 2-Sylow normalizer and the centralizer of each of its involutions by (9) and (10).

A contradiction is now apparent. Since H_1 is a proper strongly embedded subgroup of G, by Bender's theorem [1], G contains exactly one nonabelian simple composition factor $F \simeq SL(2, q)$, $Sz(q)$ or $U(3, q)$ for q a power of 2. But since J is a nonabelian simple subgroup of G, J is isomorphic to a subgroup of F. This is impossible since in the "Bender groups" centralizers of involutions are 2-closed while this subgroup-hereditary property fails in J.

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