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THE KOBAYASHI DISTANCE INDUCES THE STANDARD TOPOLOGY

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ABSTRACT. The Kobayashi pseudodistance on a connected complex space is continuous with respect to the standard topology. If this pseudodistance is an actual distance, it induces the standard topology.

Let X be a connected (reduced) complex space. The *Kobayashi pseudodistance* $d_X(p,q)$ between points p and q of X is defined as follows ([**6**, p. 462], [**7**, pp. 97–98]). Let ρ denote the distance defined by the Poincaré-Bergman metric on the open unit disk D in the complex plane. Choose points $p=p_0, p_1, \cdots, p_{k-1}, p_k=q$ of X, points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of D, and holomorphic maps $f_1, \cdots, f_k$ from D into X such that $f_j(a_j)=p_{j-1}$ and $f_j(b_j)=p_j$ for $j=1, \cdots, k$. Then $d_X(p,q)$ is defined to be the infimum of the numbers $\rho(a_1,b_1)+\cdots+\rho(a_k,b_k)$ taken over all such finite chains joining p and q. Clearly d_X is a pseudodistance on X.

In case d_X is an actual distance we will show that it induces the standard topology, i.e., the topology underlying the given complex structure on X. Several authors have tacitly used this fact; cf. [**1**, Proposition 3.8, pp. 68–69], [**4**, Proposition 1, pp. 50–51], [**5**, Theorems 2 and 3, pp. 590–591], [**6**, Theorem 3.4, pp. 465–466], [**8**, Theorem 1, pp. 11–12]. The only published proof, however, seems to be the one given by H. L. Royden [**9**, Theorem 2, pp. 133-134] for complex manifolds. Our proof uses only three elementary facts about the Kobayashi pseudodistance: if $f: Y\to Z$ is a holomorphic map, it is distance decreasing in the sense that $d_Z(f(p),f(q))\leqq d_Y(p,q)$ for all p and q in Y [**6**, Proposition 2.1, p. 462]; $d_D=\rho$ [**6**, Proposition 2.2, p. 462]; if $Q=D^n$ is a polydisk, then d_Q is continuous with respect to the standard topology on Q [7, p. 47]. In particular, we avoid Royden's differential metric.

THEOREM. *Let X be a connected complex space. Then the Kobayashi pseudodistance d_X is continuous. If d_X is an actual distance, it induces the standard topology on X.*

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PROOF. Suppose that $d_X:X\times X\to\boldsymbol{R}$ is not continuous. Since d_X satisfies the axioms for a pseudodistance, there is a point p in X for which the function $d_X(\cdot,p)$ is not continuous at p. Because X is first countable, there exist a sequence $\{p_n\}$ converging to p and a positive number ε such that $d_X(p_n,p)\geqq\varepsilon$ for all n. Hironaka's resolution of singularities [**2**, Main Theorem I′, pp. 151–152] gives an open neighborhood U of p, a complex manifold M, and a proper holomorphic map f from M onto U. (A concise explanation of the relevant part of Hironaka's terminology can be found in §1 of [**3**].) We may assume that $\{p_n\}$ is a sequence in U. Since f is onto, there is a sequence $\{q_n\}$ in M with $f(q_n)=p_n$ for all n. Because f is proper, we may, by taking a suitable subsequence if necessary, assume that $\{q_n\}$ converges to a point q of M. By continuity, $f(q)=p$. Let Q be an open neighborhood of q in M which is biholomorphically equivalent to a polydisk. Because $f|Q:Q\to X$ is distance decreasing and d_Q is continuous, $d_X(p_n,p)=d_X(f(q_n),f(q))\leqq d_Q(q_n,q)\to 0$ as $n\to\infty$. Thus $\varepsilon\leqq d_X(p_n,p)\to 0$ as $n\to\infty$, a contradiction.

Let p be a point of X, and let r be a positive real number. We will prove that the open ball $B=\{q\in X|d_X(p,q)<r\}$ is pathwise connected in X. Let q be a point of B. By the definition of the Kobayashi pseudodistance there are points $p=p_0,p_1,\cdots,p_{k-1},p_k=q$ of X, points $a_1,\cdots,a_k,b_1,\cdots,b_k$ of D, and holomorphic maps $f_1,\cdots,f_k$ from D into X such that $f_j(a_j)=p_{j-1}$ and $f_j(b_j)=p_j$ for $j=1,\cdots,n$, and $\rho(a_1,b_1)+\cdots+\rho(a_k,b_k)<r$. Let c_j denote the geodesic arc from a_j to b_j with respect to the Poincaré metric on D. Since each c_j is a geodesic arc and each f_j is distance decreasing, the path c from p to q formed by linking $f_1(c_1),\cdots,f_k(c_k)$ lies entirely in B.

Now assume that d_X is an actual distance. To prove that d_X induces the standard topology, we need only show that every open set in X is open with respect to the distance d_X. Let V be an open subset of X, and let p be a point of V. Since X is locally compact, there is a relatively compact open neighborhood W of p in X with $W\subset V$. Let r be the minimum value of the positive continuous function $d_X(p,\cdot)$ on the compact set ∂W, and let $B=\{q\in X|d_X(p,q)<r\}$. Then $B\cap\partial W=\varnothing$; since B is connected and $p\in B\cap W$, we have $B\subset W\subset V$. Thus V is open with respect to d_X. □

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