THE KOBAYASHI DISTANCE INDUCES THE STANDARD TOPOLOGY
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Abstract. The Kobayashi pseudodistance on a connected complex space is continuous with respect to the standard topology. If this pseudodistance is an actual distance, it induces the standard topology.

Let \( X \) be a connected (reduced) complex space. The Kobayashi pseudodistance \( d_X(p, q) \) between points \( p \) and \( q \) of \( X \) is defined as follows ([6, p. 462], [7, pp. 97–98]). Let \( \rho \) denote the distance defined by the Poincaré-Bergman metric on the open unit disk \( D \) in the complex plane. Choose points \( p_0 = p, p_1, \ldots, p_k = q \) of \( X \), points \( a_1, \ldots, a_k, b_1, \ldots, b_k \) of \( D \), and holomorphic maps \( f_1, \ldots, f_k \) from \( D \) into \( X \) such that \( f_j(a_j) = p_{j-1} \) and \( f_j(b_j) = p_j \) for \( j = 1, \ldots, k \). Then \( d_X(p, q) \) is defined to be the infimum of the numbers \( \rho(a_1, b_1) + \cdots + \rho(a_k, b_k) \) taken over all such finite chains joining \( p \) and \( q \). Clearly \( d_X \) is a pseudodistance on \( X \).

In case \( d_X \) is an actual distance we will show that it induces the standard topology, i.e., the topology underlying the given complex structure on \( X \). Several authors have tacitly used this fact; cf. [1, Proposition 3.8, pp. 68–69], [4, Proposition 1, pp. 50–51], [5, Theorems 2 and 3, pp. 590–591], [6, Theorem 3.4, pp. 465–466], [8, Theorem 1, pp. 11–12]. The only published proof, however, seems to be the one given by H. L. Royden [9, Theorem 2, pp. 133–134] for complex manifolds. Our proof uses only three elementary facts about the Kobayashi pseudodistance: if \( f: Y \to Z \) is a holomorphic map, it is distance decreasing in the sense that \( d_Z(f(p), f(q)) \leq d_Y(p, q) \) for all \( p \) and \( q \) in \( Y \) [6, Proposition 2.1, p. 462]; \( d_D = \rho \) [6, Proposition 2.2, p. 462]; if \( Q = D^n \) is a polydisk, then \( d_Q \) is continuous with respect to the standard topology on \( Q \) [7, p. 47]. In particular, we avoid Royden’s differential metric.

Theorem. Let \( X \) be a connected complex space. Then the Kobayashi pseudodistance \( d_X \) is continuous. If \( d_X \) is an actual distance, it induces the standard topology on \( X \).

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Proof. Suppose that \( d_X : X \times X \to \mathbb{R} \) is not continuous. Since \( d_X \) satisfies the axioms for a pseudodistance, there is a point \( p \) in \( X \) for which the function \( d_X(\cdot, p) \) is not continuous at \( p \). Because \( X \) is first countable, there exist a sequence \( \{ p_n \} \) converging to \( p \) and a positive number \( \varepsilon \) such that \( d_X(p_n, p) \geq \varepsilon \) for all \( n \). Hironaka’s resolution of singularities [2, Main Theorem I’, pp. 151–152] gives an open neighborhood \( U \) of \( p \), a complex manifold \( M \), and a proper holomorphic map \( f \) from \( M \) onto \( U \). (A concise explanation of the relevant part of Hironaka’s terminology can be found in §1 of [3].) We may assume that \( \{ q_n \} \) is a sequence in \( U \). Since \( f \) is onto, there is a sequence \( \{ q_n \} \) in \( M \) with \( f(q_n) = p_n \) for all \( n \). Because \( f \) is proper, we may, by taking a suitable subsequence if necessary, assume that \( \{ q_n \} \) converges to a point \( q \) of \( M \). By continuity, \( f(q) = p \). Let \( Q \) be an open neighborhood of \( q \) in \( M \) which is biholomorphically equivalent to a polydisk. Because \( f|Q : Q \to X \) is distance decreasing and \( d_Q \) is continuous, \( d_X(p_n, p) = d_X(f(q_n), f(q)) \leq d_Q(q_n, q) \to 0 \) as \( n \to \infty \). Thus \( \varepsilon \leq d_X(p_n, p) \to 0 \) as \( n \to \infty \), a contradiction.

Let \( p \) be a point of \( X \), and let \( r \) be a positive real number. We will prove that the open ball \( B = \{ q \in X \mid d_X(p, q) < r \} \) is pathwise connected in \( X \). Let \( q \) be a point of \( B \). By the definition of the Kobayashi pseudodistance there are points \( p = p_0, p_1, \ldots, p_k = q \) of \( X \), points \( a_1, \ldots, a_k, b_1, \ldots, b_k \) of \( D \), and holomorphic maps \( f_1, \ldots, f_k \) from \( D \) into \( X \) such that \( f_j(a_j) = p_{j-1} \) and \( f_j(b_j) = p_j \) for \( j = 1, \ldots, n \), and \( \rho(a_j, b_j) + \cdots + \rho(a_k, b_k) < r \). Let \( c_j \) denote the geodesic arc from \( a_j \) to \( b_j \) with respect to the Poincaré metric on \( D \). Since each \( c_j \) is a geodesic arc and each \( f_j \) is distance decreasing, the path \( c \) from \( p \) to \( q \) formed by linking \( f_1(c_1), \ldots, f_k(c_k) \) lies entirely in \( B \).

Now assume that \( d_X \) is an actual distance. To prove that \( d_X \) induces the standard topology, we need only show that every open set in \( X \) is open with respect to the distance \( d_X \). Let \( V \) be an open subset of \( X \), and let \( p \) be a point of \( V \). Since \( X \) is locally compact, there is a relatively compact open neighborhood \( W \) of \( p \) in \( X \) with \( W \subset V \). Let \( r \) be the minimum value of the positive continuous function \( d_X(p, \cdot) \) on the compact set \( \partial W \), and let \( B = \{ q \in X \mid d_X(p, q) < r \} \). Then \( B \cap \partial W = \emptyset \); since \( B \) is connected and \( p \in B \cap W \), we have \( B \subset W \subset V \). Thus \( V \) is open with respect to \( d_X \). □

References


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