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A NOTE ON SYSTEMS OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS

ROBERT L. WHEELER[1]

ABSTRACT. Assume the existence and boundedness of a solution to a linear system of integrodifferential equations. Conditions are found which guarantee the solution is asymptotically almost periodic.

1. **Introduction.** We consider the asymptotic behavior as $t\to\infty$ of the bounded solutions of linear systems of equations of the form

$$\int_{-\infty}^{\infty} x(t-\xi)\, dA_0(\xi) = f(t) \qquad (-\infty < t < \infty), \tag{1}$$

$$x^{(\nu)}(t) + \sum_{k=0}^{\nu-1} \int_{-\infty}^{\infty} x^{(k)}(t-\xi)\, dA_k(\xi) = f(t) \qquad (-\infty < t < \infty). \tag{2}$$

Here f and x are vectors with N components and A_k $(0 \leqq k \leqq \nu-1)$ are N by N matrices. It is assumed that $A_k \in \mathrm{NBV}(-\infty, \infty)$ for $0 \leqq k \leqq \nu-1$ (i.e. each component of A_k is of bounded variation, left-continuous and vanishes at $-\infty$), and that

$$\in L^{\infty}(-\infty, \infty), \qquad \lim_{t\to\infty} f(t) = f(\infty) \text{ exists.} \tag{1.1}$$

Let $\hat{A}(\lambda)$ denote the Fourier-Stieltjes transform

$$\hat{A}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda\xi}\, dA(\xi) \qquad (-\infty < \lambda < \infty),$$

and define the spectral sets corresponding to (1) and (2) by

$$S_1 = \{\lambda \mid \hat{A}_0(\lambda) = 0,\ -\infty < \lambda < \infty\},$$
$$S_2 = \{\lambda \mid P(i\lambda) = 0,\ -\infty < \lambda < \infty\},$$

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where $P(i\lambda)$ is the "characteristic function" associated with (2):

$$P(i\lambda) = (i\lambda)^{\nu}E + \sum_{k=0}^{\nu-1} (i\lambda)^k \hat{A}_k(\lambda) \qquad (-\infty < \lambda < \infty),$$

$$E = [\delta_{ij}], \quad \text{the } N \text{ by } N \text{ identity matrix.}$$

When the sets S_1 and S_2 are finite, Levin and Shea have shown that bounded solutions of (1) and (2) are almost periodic in a certain weak sense [**2**, Theorems 11a, 13]. They also give sufficient conditions which, for the scalar case ($N=1$), guarantee that bounded solutions of (1) and, when $\nu=1$, of (2) are asymptotically almost periodic in the sense of Fréchet [**1**], [**2**, Theorems 5c, 5a]. The purpose of this note is to extend this latter result to systems of equations as well as to systems with higher order derivatives.

For each positive integer n, consider the growth conditions.

$H(f, n)$: $f=(f_1, \cdots, f_N)$ satisfies (1.1),

$$\int_0^{\infty} t^{n-1} |f_j(t) - f_j(\infty)| \, dt < \infty \qquad (1 \leqq j \leqq N),$$

$H(A, n)$: $A=[A_{ij}] \in \mathrm{NBV}(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} |t|^n \, |dA_{ij}(t)| < \infty \qquad (1 \leqq i, j \leqq N).$$

THEOREM 1. *Let $H(f, n)$ and $H(A_0, n)$ hold, and suppose $S_1=\{\lambda_1, \cdots, \lambda_n\}$,*

$$(1.2) \qquad (d/d(i\lambda))[\det \hat{A}_0(\lambda)] \neq 0 \qquad (\lambda = \lambda_1, \cdots, \lambda_n).$$

Let $x(t)$ be a bounded, Borel measurable function which satisfies (1) *on* $(-\infty, \infty)$ *as well as the tauberian condition*

$$(\mathrm{T}) \qquad \lim_{t\to\infty, \tau\to 0} |x(t+\tau) - x(t)| = 0.$$

Then

$$(1.3) \quad x(t) = f(\infty)A_0(\infty)^{-1} + \sum_{m=1}^{n} \gamma_m \exp[i\lambda_m t] + \eta(t) \qquad (-\infty < t < \infty)$$

where $\gamma_m \in C^N$ $(1\leqq m\leqq n)$ and $\eta(t)\to 0$ as $t\to\infty$ (the term $f(\infty)A_0(\infty)^{-1}$ does not appear in (1.3) *if one of the $\lambda_m=0$).*

The analogous theorem for (2) is

THEOREM 2. *Let $H(f, n)$, $H(A_k, n)$ $(0\leqq k\leqq \nu-1)$ hold, and assume $S_2=\{\lambda_1, \cdots, \lambda_n\}$,*

$$(1.4) \qquad (d/d(i\lambda))[\det P(i\lambda)] \neq 0 \qquad (\lambda = \lambda_1, \cdots, \lambda_n).$$

Let $x(t) \in L^\infty(-\infty, \infty)$ *with* $x^{(\nu-1)}(t)$ *locally absolutely continuous* (*LAC*) *on* $(-\infty, \infty)$ *satisfy* (2) *a.e. on* $(-\infty, \infty)$. *Suppose, in addition, that*

$$x^{(k)}(t) \in L^\infty(-\infty, \infty) \qquad (1 \leqq k \leqq \nu - 1). \tag{1.5}$$

Then (1.3) *holds with* η *satisfying*

$$\lim_{t\to\infty} \eta^{(k)}(t) = 0 \quad (0 \leqq k \leqq \nu - 1), \qquad \lim_{t\to\infty} \left\{ \operatorname*{ess\,sup}_{t<\tau<\infty} |\eta^{(\nu)}(\tau)| \right\} = 0. \tag{1.6}$$

We remark that the derivatives in (1.2) and (1.4) always exist whenever $H(A_k, n)$ $(0 \leqq k \leqq \nu-1)$ hold.

By a simple conversion process [2, Lemma 19.2] we may use Theorems 1 and 2 to obtain analogous results about the corresponding Volterra equations

$$\int_0^t x(t-\xi)\, dA_0(\xi) = f(t) \qquad (0 \leqq t < \infty), \tag{1'}$$

$$x^{(\nu)}(t) + \sum_{k=0}^{\nu-1} \int_0^t x^{(k)}(t-\xi)\, dA_k(\xi) = f(t) \qquad (0 \leqq t < \infty). \tag{2'}$$

In the case of (2′), we need not assume *a priori* (the analogue for $[0, \infty)$ of) hypothesis (1.5) since this will follow from the other hypotheses [2, Lemma 19.1].

COROLLARY. *Let* $x(t) \in L^\infty(0, \infty)$ (*with* $x^{(\nu-1)}(t) \in LAC[0, \infty)$) *satisfy* (2′) *a.e. on* $[0, \infty)$ *with* $A_k = [A_{kij}] \in NBV[0, \infty)$, *and* $f \in L^\infty(0, \infty)$ *such that* $\lim_{t\to\infty} f(t) = f(\infty)$ *exists. Define* S_2 *and* $P(i\lambda)$ *as before where the* $A_k(t)$ *are understood to be identically zero on* $(-\infty, 0]$. *Suppose* $S_2 = \{\lambda_1, \cdots, \lambda_n\}$, *and* (1.4),

$$\int_0^\infty t^n\, |dA_{kij}(t)| < \infty \qquad (0 \leqq k \leqq \nu - 1,\ 1 \leqq i, j \leqq N), \tag{1.7}$$

$$\int_0^\infty t^{n-1}\, |f_j(t) - f_j(\infty)|\, dt < \infty \qquad (1 \leqq j \leqq N) \tag{1.8}$$

are satisfied. Then (1.3) *holds on* $[0, \infty)$ *with* η *satisfying* (1.6).

When $n=1$, it is easy to see that (1.8) cannot be omitted from the hypotheses of this Corollary by observing that

$$x(t) = \frac{1}{(\nu-1)!} \int_0^t (1 - e^{\tau-t})^{\nu-1} f(\tau)\, d\tau + x(0) \qquad (0 \leqq t < \infty)$$

is a solution of the differential equation

$$x^{(\nu)}(t) + \sum_{k=0}^{\nu-1} \alpha_k x^{(k)}(t) = f(t) \qquad (0 \leqq t < \infty), \tag{1.9}$$

where the constants α_k are chosen so that the characteristic polynomial of (1.9) is $P(z)=\prod_{k=0}^{\nu-1}(z+k)$. This is a special case of (2′) with $N=1$, $A_k(0)=0$, $A_k(t)=\alpha_k$ $(t>0,\ 0\leqq k\leqq \nu-1)$.

2. **Proof of Theorem 1.** We may assume $N\geqq 2$ since the case $N=1$ is [**2**, Theorem 5c]. Let $x\star A$ denote the convolution

$$x\star A(t)=\int_{-\infty}^{\infty} x(t-\xi)\,dA(\xi) \qquad (-\infty<t<\infty),$$

that is $x\star A=(z_1,\cdots,z_N)$ where $z_j(t)=\sum_{i=1}^{N}\int_{-\infty}^{\infty} x_i(t-\xi)\,dA_{ij}(\xi)$. By (1),

$$x\star A_0\star \operatorname{adj} A_0(t)=f\star \operatorname{adj} A_0(t) \qquad (-\infty<t<\infty),$$

where adj A_0 denotes the N by N matrix obtained by taking the formal adjoint of A_0, but with convolution replacing pointwise multiplication. This equation may be rewritten as N scalar equations

$$x_j\star B(t)=h_j(t) \qquad (1\leqq j\leqq N,\ -\infty<t<\infty), \tag{2.1}$$

where $B\in \mathrm{NBV}(-\infty,\infty)$ is the scalar function defined by taking the formal determinant of A_0 in which pointwise multiplication is replaced by convolution, and $h=(h_1,\cdots,h_N)=f\star \operatorname{adj} A_0$. Using $H(A_0,n)$ and $H(f,n)$, one easily verifies that $H(B,n)$ and $H(h,n)$ hold with $\lim_{t\to\infty} h(t)=f(\infty)(\operatorname{adj} A_0)(\infty)$. Since $\hat{B}(\lambda)=\det[\hat{A}_0(\lambda)]$ $(-\infty<\lambda<\infty)$, the scalar case of Theorem 1 may be applied to each equation in (2.1) to yield

$$x_j(t)=h_j(\infty)B(\infty)^{-1}+\sum_{m=1}^{n}\gamma_{mj}\exp[i\lambda_m t]+\eta_j(t)$$
$$(1\leqq j\leqq N,\ -\infty<t<\infty)$$

with $\gamma_{mj}\in \boldsymbol{C}$ and $\eta_j(t)\to 0$ as $t\to\infty$ (the terms $h_j(\infty)B(\infty)^{-1}$ do not occur if one of the $\lambda_m=0$). Theorem 1 follows by setting $\gamma_m=(\gamma_{m1},\cdots,\gamma_{mN})$ for $1\leqq m\leqq n$ and $\eta=(\eta_1,\cdots,\eta_N)$.

3. **Proof of Theorem 2.** We deduce Theorem 2 from Theorem 1. Let

$$G(t)=\int_{-\infty}^{t}\exp[-\xi^2]\,d\xi \qquad (-\infty<t<\infty),$$

so that $H(G,n)$, $H(G',n)$ hold for any positive integer n, and

$$\hat{G}(\lambda)\neq 0,\quad (G')^{\wedge}(\lambda)=(i\lambda)\hat{G}(\lambda) \qquad (-\infty<\lambda<\infty). \tag{3.1}$$

Using $x*a$ to denote $x*a(t)=\int_{-\infty}^{\infty} x(t-\xi)a(\xi)\,d\xi$, (2) gives

$$x_j^{(\nu)}*G'(t)+\sum_{k=0}^{\nu-1}\sum_{i=1}^{N} x_i^{(k)}\star A_{kij}*G'(t)=f_j*G'(t)$$
$$(1\leqq j\leqq N,\ -\infty<t<\infty)$$

where $A_k = [A_{kij}]$. Integrating by parts yields

$$x_j^{(\nu-1)} \star G'(t) + \sum_{k=1}^{\nu-1} \sum_{i=1}^{N} x_i^{(k-1)} \star A_{kij} \star G'(t) + \sum_{i=1}^{N} x_i \star A_{0ij} \star G(t)$$
$$= f_j * G'(t) \qquad (1 \leqq j \leqq N, \ -\infty < t < \infty).$$

Repeating this convolution and integration process ν times, we obtain

$$x \star B(t) = g(t) \qquad (-\infty < t < \infty) \tag{3.2}$$

where $B = [B_{ij}]$ with

$$B_{ij} = \delta_{ij}(G')^{\nu\star} + \sum_{k=1}^{\nu-1} A_{kij} \star (G')^{k\star} \star G^{(\nu-k)\star} + A_{0ij} \star G^{\nu\star},$$

δ_{ij} denoting the Kronecker delta, and $g = f * C$ where $C = [C_{ij}]$ with $C_{ij} = \delta_{ij}(G')^{\nu *}$. Here we have used $A^{k\star}$ and a^{k*} to denote the k-fold convolutions $A \star A \star \cdots \star A$, $a * a * \cdots * a$. Using $H(G, n)$, $H(G', n)$, $H(A_k, n)$ $(0 \leqq k \leqq \nu - 1)$ and $H(f, n)$, one easily checks that $H(B, n)$ and $H(g, n)$ are satisfied with $\lim_{t\to\infty} g(t) = f(\infty)[(G(\infty))^\nu E]$. The definition of B and (3.1) imply

$$\det \hat{B}(\lambda) = [\hat{G}(\lambda)^{(N\nu)} \det[P(i\lambda)] \qquad (-\infty < \lambda < \infty);$$

hence (2) and (3.2) have the same spectral sets. Also, x satisfies (T) since $x \in \mathrm{LAC}(-\infty, \infty)$ and $\|x'\|_\infty < \infty$. Thus, applying Theorem 1 to (3.2) and using the values of $g(\infty)$ and $B(\infty)$, we find (1.3) holds with $\gamma_m \in \boldsymbol{C}^N$ $(1 \leqq m \leqq n)$ and $\eta(t) \to 0$ as $t \to \infty$. To show η satisfies (1.6), note that (the analogue for $(-\infty, \infty)$ of) [**2**, Theorem 13a] implies

$$x(t) = f(\infty)A_0(\infty)^{-1} + \sum_{m=1}^{n} c_m(t)\exp[i\lambda_m t] + \eta_1(t) \qquad (-\infty < t < \infty)$$

where η_1 satisfies (1.6), and the c_m satisfy

$$c_m \in C^\infty(-\infty, \infty) \cap L^\infty(-\infty, \infty), \quad \lim_{t\to\infty} c_m^{(j)}(t) = 0 \qquad (j = 1, 2, \cdots). \tag{3.3}$$

Since $\sum_{m=1}^{n} (c_m(t) - \gamma_m)\exp[i\lambda_m t] \to 0$ as $t \to \infty$, it follows from (3.3) that $c_m(t) \to \gamma_m$ as $t \to \infty$ $(1 \leqq m \leqq n)$. Thus η satisfies (1.6).

We remark that the technique used to prove Theorem 2 may also be employed to give a different proof of Theorem 13a in [**2**].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA, MISSOURI 65201