A NOTE ON SYSTEMS OF LINEAR INTEGRODIFFERENTIAL EQUATIONS

ROBERT L. WHEELER

Abstract. Assume the existence and boundedness of a solution to a linear system of integrodifferential equations. Conditions are found which guarantee the solution is asymptotically almost periodic.

1. Introduction. We consider the asymptotic behavior as \( t \to \infty \) of the bounded solutions of linear systems of equations of the form

\[
\begin{align*}
(1) & \quad \int_{-\infty}^{\infty} x(t - \xi) \, dA_0(\xi) = f(t) \quad (-\infty < t < \infty), \\
(2) & \quad x^{(v)}(t) + \sum_{k=0}^{v-1} \int_{-\infty}^{\infty} x^{(k)}(t - \xi) \, dA_k(\xi) = f(t) \quad (-\infty < t < \infty).
\end{align*}
\]

Here \( f \) and \( x \) are vectors with \( N \) components and \( A_k \) \((0 \leq k \leq v-1)\) are \( N \) by \( N \) matrices. It is assumed that \( A_k \in NBV(-\infty, \infty) \) for \( 0 \leq k \leq v-1 \) (i.e. each component of \( A_k \) is of bounded variation, left-continuous and vanishes at \( -\infty \)), and that

\[
(1.1) \quad \in L^\infty(-\infty, \infty), \quad \lim_{t \to \infty} f(t) = f(\infty) \text{ exists}.
\]

Let \( \hat{A}(\lambda) \) denote the Fourier-Stieltjes transform

\[
\hat{A}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda \xi} \, dA(\xi) \quad (-\infty < \lambda < \infty),
\]

and define the spectral sets corresponding to (1) and (2) by

\[
S_1 = \{ \lambda \mid \hat{A}_0(\lambda) = 0, \, -\infty < \lambda < \infty \}, \quad S_2 = \{ \lambda \mid P(i\lambda) = 0, \, -\infty < \lambda < \infty \}.
\]
where \( P(i\lambda) \) is the "characteristic function" associated with (2):
\[
P(i\lambda) = (i\lambda)^v E + \sum_{k=0}^{v-1} (i\lambda)^k \hat{A}_k(\lambda) \quad (-\infty < \lambda < \infty),
\]
\[E = [\delta_{ij}], \quad \text{the } N \text{ by } N \text{ identity matrix.}
\]

When the sets \( S_1 \) and \( S_2 \) are finite, Levin and Shea have shown that bounded solutions of (1) and (2) are almost periodic in a certain weak sense [2, Theorems 11a, 13]. They also give sufficient conditions which, for the scalar case \((N=1)\), guarantee that bounded solutions of (1) and, when \( v=1 \), of (2) are asymptotically almost periodic in the sense of Fréchet [1], [2, Theorems 5c, 5a]. The purpose of this note is to extend this latter result to systems of equations as well as to systems with higher order derivatives.

For each positive integer \( n \), consider the growth conditions.

\[
H(f, n): f = (f_1, \ldots, f_n) \text{ satisfies } (1.1),
\]
\[
\int_0^\infty t^{n-1} |f_j(t) - f_j(\infty)| \, dt < \infty \quad (1 \leq j \leq N),
\]

\[
H(A, n): A = [A_{ij}] \in \text{NBV}(-\infty, \infty),
\]
\[
\int_{-\infty}^\infty |t^n| |dA_{ij}(t)| < \infty \quad (1 \leq i, j \leq N).
\]

**Theorem 1.** Let \( H(f, n) \) and \( H(A_0, n) \) hold, and suppose \( S_1 = \{\lambda_1, \ldots, \lambda_n\}, \)
\[
(1.2) \quad (d/d(i\lambda)) [\det \hat{A}_0(\lambda)] \neq 0 \quad (\lambda = \lambda_1, \ldots, \lambda_n).
\]

Let \( x(t) \) be a bounded, Borel measurable function which satisfies (1) on \((-\infty, \infty)\) as well as the tauberian condition
\[\lim_{t \to \infty, \tau \to 0} |x(t + \tau) - x(t)| = 0.\]
Then
\[
x(t) = f(\infty)A_0(\infty)^{-1} + \sum_{m=1}^{n} \gamma_m \exp[i\lambda_m t] + \eta(t) \quad (-\infty < t < \infty)
\]
where \( \gamma_m \in C^N \) \((1 \leq m \leq n)\) and \( \eta(t) \to 0 \) as \( t \to \infty \) (the term \( f(\infty)A_0(\infty)^{-1} \) does not appear in (1.3) if one of the \( \lambda_m = 0 \)).

The analogous theorem for (2) is

**Theorem 2.** Let \( H(f, n), \ H(A_k, n) \) \((0 \leq k \leq v-1)\) hold, and assume \( S_k = \{\lambda_1, \ldots, \lambda_n\}, \)
\[
(1.4) \quad (d/d(i\lambda)) [\det P(i\lambda)] \neq 0 \quad (\lambda = \lambda_1, \ldots, \lambda_n).
\]
Let \( x(t) \in L^\infty(-\infty, \infty) \) with \( x^{(v-1)}(t) \) locally absolutely continuous (LAC) on \((-\infty, \infty)\) satisfy (2) a.e. on \((-\infty, \infty)\). Suppose, in addition, that

\[
(1.5) \quad x^{(k)}(t) \in L^\infty(-\infty, \infty) \quad (1 \leq k \leq v - 1).
\]

Then (1.3) holds with \( \eta \) satisfying

\[
(1.6) \quad \lim_{t \to \infty} \eta^{(k)}(t) = 0 \quad (0 \leq k \leq v - 1), \quad \lim_{t \to \infty} \left( \text{ess sup} |\eta^{(v)}(t)| \right) = 0.
\]

We remark that the derivatives in (1.2) and (1.4) always exist whenever \( H(A_k, n) \) (0 \( k \leq \nu - 1 \)) hold.

By a simple conversion process [2, Lemma 19.2] we may use Theorems 1 and 2 to obtain analogous results about the corresponding Volterra equations

\[
(1') \quad \int_0^t x(t - \xi) \, dA_k(\xi) = f(t) \quad (0 \leq t < \infty),
\]

\[
(2') \quad x^{(v)}(t) + \sum_{k=0}^{v-1} \int_0^t x^{(k)}(t - \xi) \, dA_k(\xi) = f(t) \quad (0 \leq t < \infty).
\]

In the case of (2'), we need not assume a priori (the analogue for [0, \( \infty \)) of) hypothesis (1.5) since this will follow from the other hypotheses [2, Lemma 19.1].

**Corollary.** Let \( x(t) \in L^\infty(0, \infty) \) (with \( x^{(v-1)}(t) \in L^\infty([0, \infty]) \)) satisfy (2') a.e. on [0, \( \infty \)) with \( A_k = [A_{ki}] \in NBV[0, \infty) \), and \( f \in L^\infty(0, \infty) \) such that \( \lim_{t \to \infty} f(t) = f(\infty) \) exists. Define \( S_2 \) and \( P(\nu) \) as before where the \( A_k(t) \) are understood to be identically zero on \((-\infty, 0)\). Suppose \( S_2 = \{\lambda_1, \cdots, \lambda_n\} \), and (1.4),

\[
(1.7) \quad \int_0^\infty t^k |dA_{ki}(t)| < \infty \quad (0 \leq k \leq \nu - 1, 1 \leq i, j \leq N),
\]

\[
(1.8) \quad \int_0^\infty t^{j-1} |f_j(t) - f_j(\infty)| \, dt < \infty \quad (1 \leq j \leq N)
\]

are satisfied. Then (1.3) holds on [0, \( \infty \)) with \( \eta \) satisfying (1.6).

When \( n=1 \), it is easy to see that (1.8) cannot be omitted from the hypotheses of this Corollary by observing that

\[
x(t) = \frac{1}{(\nu - 1)!} \int_0^t (1 - e^{\nu t})^{v-1} f(\tau) \, d\tau + x(0) \quad (0 \leq t < \infty)
\]

is a solution of the differential equation

\[
(1.9) \quad x^{(v)}(t) + \sum_{k=0}^{v-1} a_k x^{(k)}(t) = f(t) \quad (0 \leq t < \infty),
\]
where the constants \(a_k\) are chosen so that the characteristic polynomial of (1.9) is \(P(z) = \prod_{k=0}^{\nu} (z+k)\). This is a special case of (2') with \(N=1\), \(A_k(0)=0\), \(A_k(t) = a_k(t > 0, 0 \leq k \leq \nu - 1)\).

2. Proof of Theorem 1. We may assume \(N \geq 2\) since the case \(N=1\) is [2, Theorem 5c]. Let \(x \star A\) denote the convolution

\[
x \star A(t) = \int_{-\infty}^{\infty} x(t - \xi) dA(\xi) \quad (-\infty < t < \infty),
\]

that is \(x \star A = (z_1, \ldots, z_N)\) where \(z_j(t) = \sum_{i=1}^{N} \int_{-\infty}^{\infty} x_i(t - \xi) dA_{ij}(\xi)\). By (1),

\[
x \star A_0 \star \text{adj} \ A_0(t) = f \star \text{adj} \ A_0(t) \quad (-\infty < t < \infty),
\]

where \(\text{adj} \ A_0\) denotes the \(N\) by \(N\) matrix obtained by taking the formal adjoint of \(A_0\), but with convolution replacing pointwise multiplication. This equation may be rewritten as \(N\) scalar equations

\[
x_j \star B(t) = h_j(t) \quad (1 \leq j \leq N, -\infty < t < \infty),
\]

where \(B \in \text{NBV}(\infty, \infty)\) is the scalar function defined by taking the formal determinant of \(A_0\) in which pointwise multiplication is replaced by convolution, and \(h = (h_1, \ldots, h_N) = f \star \text{adj} \ A_0\). Using \(H(A_0, n)\) and \(H(f, n)\), one easily verifies that \(H(B, n)\) and \(H(h, n)\) hold if \(\lim_{t \to \infty} h(t) = f(\infty)(\text{adj} \ A_0)(\infty)\). Since \(B(\lambda) = \det[A_0(\lambda)]\) \((-\infty < \lambda < \infty)\), the scalar case of Theorem 1 may be applied to each equation in (2.1) to yield

\[
x_j(t) = h_j(\infty)B(\infty)^{-1} + \sum_{m=1}^{n} \gamma_m(t) \exp[i\lambda_m t] + \eta_j(t) \quad (1 \leq j \leq N, -\infty < t < \infty)
\]

with \(\gamma_m \in C\) and \(\eta_j(t) \to 0\) as \(t \to \infty\) (the terms \(h_j(\infty)B(\infty)^{-1}\) do not occur if one of the \(\lambda_m = 0\)). Theorem 1 follows by setting \(\gamma_m = (\gamma_{m1}, \ldots, \gamma_{mN})\) for \(1 \leq m \leq n\) and \(\eta = (\eta_1, \ldots, \eta_N)\).

3. Proof of Theorem 2. We deduce Theorem 2 from Theorem 1. Let

\[
G(t) = \int_{-\infty}^{t} \exp[-\xi^2] d\xi \quad (-\infty < t < \infty),
\]

so that \(H(G, n), H(G', n)\) hold for any positive integer \(n\), and

\[
\tilde{G}(\lambda) \neq 0, \quad (G')^\wedge(\lambda) = (i\lambda)G(\lambda) \quad (-\infty < \lambda < \infty).
\]

Using \(x \star a\) to denote \(x \star a(t) = \int_{-\infty}^{\infty} x(t - \xi)a(\xi) d\xi\), (2) gives

\[
x_j^{(v)} \star G'(t) + \sum_{k=0}^{v-1} \sum_{i=1}^{N} x_i^{(k)} \star A_{kij} \star G'(t) = f_j \star G'(t) \quad (1 \leq j \leq N, -\infty < t < \infty)
\]
where $A_k = [A_{kij}]$. Integrating by parts yields

$$x_j^{(v-1)} \ast G'(t) + \sum_{k=1}^{v-1} \sum_{i=1}^{N} x_i^{(k-1)} \ast A_{kij} \ast G'(t) + \sum_{i=1}^{N} x_i \ast A_{0ij} \ast G(t) = f_j \ast G'(t) \quad (1 \leq j \leq N, -\infty < t < \infty).$$

Repeating this convolution and integration process $v$ times, we obtain

$$(3.2) \quad x \ast B(t) = g(t) \quad (-\infty < t < \infty)$$

where $B = [B_{ij}]$ with

$$B_{ij} = \delta_{ij} (G')^v + \sum_{k=1}^{v-1} A_{kij} \ast (G')^{k} \ast G^{(v-k)+} + A_{0ij} \ast G^v,$$

$\delta_{ij}$ denoting the Kronecker delta, and $g = f \ast C$ where $C = [C_{ij}]$ with $C_{ij} = \delta_{ij} (G')^v$. Here we have used $A^v$ and $a^v$ to denote the $k$-fold convolutions $A \ast A \ast \cdots \ast A$, $a \ast a \ast \cdots \ast a$. Using $H(G, n)$, $H(G', n)$, $H(A_k, n)$ ($0 \leq k \leq v-1$) and $H(f, n)$, one easily checks that $H(B, n)$ and $H(g, n)$ are satisfied with $\lim_{t \to \infty} g(t) = f(\infty)(G(\infty))^vE$. The definition of $B$ and (3.1) imply

$$\det \hat{B}(\lambda) = \det[\hat{G}(\lambda)^v] \det[P(i\lambda)] \quad (-\infty < \lambda < \infty);$$

hence (2) and (3.2) have the same spectral sets. Also, $x$ satisfies (T) since $x \in \text{LAC}(-\infty, \infty)$ and $\|x'\|_{\infty} < \infty$. Thus, applying Theorem 1 to (3.2) and using the values of $g(\infty)$ and $B(\infty)$, we find (1.3) holds with $\gamma_m \in C^N$ ($1 \leq m \leq n$) and $\eta(t) \to 0$ as $t \to \infty$. To show $\eta$ satisfies (1.6), note that (the analogue for $(-\infty, \infty)$ of) [2, Theorem 13a] implies

$$x(t) = f(\infty)A_0(\infty)^{-1} + \sum_{m=1}^{n} c_m(t) \exp[i\lambda_m t] + \eta_1(t) \quad (-\infty < t < \infty)$$

where $\eta_1$ satisfies (1.6), and the $c_m$ satisfy

$$(3.3) \quad c_m \in C^\infty(-\infty, \infty) \cap L^\infty(-\infty, \infty), \quad \lim_{t \to \infty} c_m^{(j)}(t) = 0 \quad (j = 1, 2, \cdots).$$

Since $\sum_{m=1}^{n} (c_m(t) - \gamma_m) \exp[i\lambda_m t] \to 0$ as $t \to \infty$, it follows from (3.3) that $c_m(t) \to \gamma_m$ as $t \to \infty$ ($1 \leq m \leq n$). Thus $\eta$ satisfies (1.6).

We remark that the technique used to prove Theorem 2 may also be employed to give a different proof of Theorem 13a in [2].
REFERENCES


Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65201