A NOTE ON SYSTEMS OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS

ROBERT L. WHEELER

Abstract. Assume the existence and boundedness of a solution to a linear system of integrodifferential equations. Conditions are found which guarantee the solution is asymptotically almost periodic.

1. Introduction. We consider the asymptotic behavior as \( t \to \infty \) of the bounded solutions of linear systems of equations of the form

\[
\begin{align*}
\text{(1)} \quad & x(t) = \int_{-\infty}^{\infty} x(t - \xi) \, dA_0(\xi) = f(t) \quad (-\infty < t < \infty), \\
\text{(2)} \quad & x^{(v)}(t) + \sum_{k=0}^{v-1} x^{(k)}(t - \xi) \, dA_k(\xi) = f(t) \quad (-\infty < t < \infty).
\end{align*}
\]

Here \( f \) and \( x \) are vectors with \( N \) components and \( A_k \) \( (0 \leq k \leq v-1) \) are \( N \) by \( N \) matrices. It is assumed that \( A_k \in \text{NBV}(-\infty, \infty) \) for \( 0 \leq k \leq v-1 \) (i.e. each component of \( A_k \) is of bounded variation, left-continuous and vanishes at \( -\infty \)), and that

\[
\text{(1.1)} \quad \in L^\infty(-\infty, \infty), \quad \lim_{t \to \infty} f(t) = f(\infty) \text{ exists.}
\]

Let \( \hat{A}(\lambda) \) denote the Fourier-Stieltjes transform

\[
\hat{A}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda \xi} \, dA(\xi) \quad (-\infty < \lambda < \infty),
\]

and define the spectral sets corresponding to (1) and (2) by

\[
S_1 = \{ \lambda \mid \hat{A}_0(\lambda) = 0, \ -\infty < \lambda < \infty \},
\]

\[
S_2 = \{ \lambda \mid P(i\lambda) = 0, \ -\infty < \lambda < \infty \},
\]

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where \( P(i\lambda) \) is the "characteristic function" associated with (2):

\[
P(i\lambda) = (i\lambda)^n E + \sum_{k=0}^{v-1} (i\lambda)^k \tilde{A}_k(\lambda) \quad (-\infty < \lambda < \infty),
\]

\[
E = [\delta_{ij}], \quad \text{the } N \text{ by } N \text{ identity matrix.}
\]

When the sets \( S_1 \) and \( S_2 \) are finite, Levin and Shea have shown that bounded solutions of (1) and (2) are almost periodic in a certain weak sense [2, Theorems 11a, 13]. They also give sufficient conditions which, for the scalar case \((N=1)\), guarantee that bounded solutions of (1) and, when \( v=1 \), of (2) are asymptotically almost periodic in the sense of Fréchet [1], [2, Theorems 5c, 5a]. The purpose of this note is to extend this latter result to systems of equations as well as to systems with higher order derivatives.

For each positive integer \( n \), consider the growth conditions.

\( H(f, n) \): \( f = (f_1, \ldots, f_N) \) satisfies (1.1),

\[
\int_{-\infty}^{\infty} t^{n-1} \left| f_j(t) - f_j(\infty) \right| dt < \infty \quad (1 \leq j \leq N),
\]

\( H(A, n) \): \( A = [A_{ij}] \in \text{NBV}(-\infty, \infty), \)

\[
\int_{-\infty}^{\infty} |t|^n |dA_{ij}(t)| < \infty \quad (1 \leq i, j \leq N).
\]

**Theorem 1.** Let \( H(f, n) \) and \( H(A, n) \) hold, and suppose \( S_1 = \{\lambda_1, \ldots, \lambda_n\}, \)

\[
(1.2) \quad (d/d(i\lambda))[\det \tilde{A}_0(\lambda)] \neq 0 \quad (\lambda = \lambda_1, \ldots, \lambda_n).
\]

Let \( x(t) \) be a bounded, Borel measurable function which satisfies (1) on \((-\infty, \infty)\) as well as the tauberian condition

\[
(T) \quad \lim_{t \to \infty, r \to 0} \left| x(t + r) - x(t) \right| = 0.
\]

Then

\[
x(t) = f(\infty) A_0(\infty)^{-1} + \sum_{m=1}^{n} \gamma_m \exp[i\lambda_m t] + \eta(t) \quad (-\infty < t < \infty)
\]

where \( \gamma_m \in C^N \) \((1 \leq m \leq n)\) and \( \eta(t) \to 0 \) as \( t \to \infty \) (the term \( f(\infty) A_0(\infty)^{-1} \) does not appear in (1.3) if one of the \( \lambda_m=0 \)).

The analogous theorem for (2) is

**Theorem 2.** Let \( H(f, n) \), \( H(A_k, n) \) \((0 \leq k \leq v-1)\) hold, and assume \( S_2 = \{\lambda_1, \ldots, \lambda_n\}, \)

\[
(1.4) \quad (d/d(i\lambda))[\det P(i\lambda)] \neq 0 \quad (\lambda = \lambda_1, \ldots, \lambda_n).
\]
Let \( x(t) \in L^\infty(-\infty, \infty) \) with \( x^{(v-1)}(t) \) locally absolutely continuous (LAC) on \((-\infty, \infty)\) satisfy (2) a.e. on \((-\infty, \infty)\). Suppose, in addition, that

\[
(1.5) \quad x^{(k)}(t) \in L^\infty(-\infty, \infty) \quad (1 \leq k \leq v - 1).
\]

Then (1.3) holds with \( \eta \) satisfying

\[
(1.6) \quad \lim_{t \to \infty} \eta^{(k)}(t) = 0 \quad (0 \leq k \leq v - 1), \quad \lim_{t \to \infty} \left( \operatorname{ess \ sup}_{t < r < \infty} |\eta^{(v)}(r)| \right) = 0.
\]

We remark that the derivatives in (1.2) and (1.4) always exist whenever \( H(A_k, n) \) (0 \( \leq k \leq v - 1 \)) hold.

By a simple conversion process [2, Lemma 19.2] we may use Theorems 1 and 2 to obtain analogous results about the corresponding Volterra equations

\[
(1') \quad \int_0^t x(t - \xi) \, dA_k(\xi) = f(t) \quad (0 \leq t < \infty),
\]

\[
(2') \quad x^{(v)}(t) + \sum_{k=0}^{v-1} \int_0^t x^{(k)}(t - \xi) \, dA_k(\xi) = f(t) \quad (0 \leq t < \infty).
\]

In the case of (2'), we need not assume a priori (the analogue for \([0, \infty)\) of) hypothesis (1.5) since this will follow from the other hypotheses [2, Lemma 19.1].

**Corollary.** Let \( x(t) \in L^\infty(0, \infty) \) (with \( x^{(v-1)}(t) \in LAC[0, \infty) \)) satisfy (2') a.e. on \([0, \infty)\) with \( A_k = [A_{kj}] \in NBV[0, \infty) \), and \( f \in L^\infty(0, \infty) \) such that \( \lim_{t \to \infty} f(t) = f(\infty) \) exists. Define \( S_2 \) and \( P(\mu) \) as before where the \( A_k(t) \) are understood to be identically zero on \((-\infty, 0]\). Suppose \( S_2 = \{\lambda_1, \ldots, \lambda_n\} \), and (1.4),

\[
(1.7) \quad \int_0^\infty t^n |dA_{kj}(t)| < \infty \quad (0 \leq k \leq v - 1, 1 \leq i, j \leq N),
\]

\[
(1.8) \quad \int_0^\infty t^{n-1} |f_j(t) - f_j(\infty)| \, dt < \infty \quad (1 \leq j \leq N)
\]

are satisfied. Then (1.3) holds on \([0, \infty)\) with \( \eta \) satisfying (1.6).

When \( n=1 \), it is easy to see that (1.8) cannot be omitted from the hypotheses of this Corollary by observing that

\[
x(t) = \frac{1}{(v - 1)} \int_0^t \left( 1 - e^{-\tau} \right)^{v-1} f(\tau) \, d\tau + x(0) \quad (0 \leq t < \infty)
\]

is a solution of the differential equation

\[
(1.9) \quad x^{(v)}(t) + \sum_{k=0}^{v-1} a_k x^{(k)}(t) = f(t) \quad (0 \leq t < \infty),
\]
where the constants $\alpha_k$ are chosen so that the characteristic polynomial of (1.9) is $P(z) = \prod_{k=0}^{v-1} (z + k)$. This is a special case of (2') with $N = 1$, $A_k(0) = 0$, $A_k(t) = \alpha_k$ ($t > 0$, $0 \leq k \leq v - 1$).

2. Proof of Theorem 1. We may assume $N \geq 2$ since the case $N = 1$ is [2, Theorem 5c]. Let $x * A$ denote the convolution

$$x * A(t) = \int_{-\infty}^{\infty} x(t - \xi) dA(\xi), \quad (-\infty < t < \infty),$$

that is $x * A = (z_1, \cdots, z_N)$ where $z_j(t) = \sum_{i=1}^{N} \int_{-\infty}^{\infty} x_i(t - \xi) dA_{ij}(\xi)$. By (1),

$$x * A_0 * \text{adj } A_0(t) = f * \text{adj } A_0(t), \quad (-\infty < t < \infty),$$

where $\text{adj } A_0$ denotes the $N$ by $N$ matrix obtained by taking the formal adjoint of $A_0$, but with convolution replacing pointwise multiplication. This equation may be rewritten as $N$ scalar equations

$$x_j * B(t) = h_j(t), \quad (1 \leq j \leq N, -\infty < t < \infty),$$

where $B \in \text{NBV}(-\infty, \infty)$ is the scalar function defined by taking the formal determinant of $A_0$ in which pointwise multiplication is replaced by convolution, and $h = (h_1, \cdots, h_N) = f * \text{adj } A_0$. Using $H(A_0, n)$ and $H(f, n)$, one easily verifies that $H(B, n)$ holds when $\lim_{t \to \infty} h(t) = f(\infty)(\text{adj } A_0)(\infty)$. Since $B(\lambda) = \det[A_0(\lambda)]$ ($-\infty < \lambda < \infty$), the scalar case of Theorem 1 may be applied to each equation in (2.1) to yield

$$x_j(t) = h_j(\infty)B(\infty)^{-1} + \sum_{m=1}^{n} \gamma_m \exp[i\lambda_m t] + \eta_j(t)$$

$(1 \leq j \leq N, -\infty < t < \infty)$

with $\gamma_m \in C$ and $\eta_j(t) \to 0$ as $t \to \infty$ (the terms $h_j(\infty)B(\infty)^{-1}$ do not occur if one of the $\lambda_m = 0$). Theorem 1 follows by setting $\gamma_m = (\gamma_{m1}, \cdots, \gamma_{mN})$ for $1 \leq m \leq n$ and $\eta = (\eta_1, \cdots, \eta_N)$.

3. Proof of Theorem 2. We deduce Theorem 2 from Theorem 1. Let

$$G(t) = \int_{-\infty}^{t} \exp[-\xi^2] d\xi, \quad (-\infty < t < \infty),$$

so that $H(G, n)$, $H(G', n)$ hold for any positive integer $n$, and

$$\hat{G}(\lambda) \neq 0, \quad (G')^\wedge(\lambda) = (i\lambda)\hat{G}(\lambda) \quad (-\infty < \lambda < \infty).$$

Using $x * a$ to denote $x * a(t) = \int_{-\infty}^{\infty} x(t - \xi) a(\xi) d\xi$, (2) gives

$$x_{j}^{(v)} * G'(t) + \sum_{k=0}^{v-1} \sum_{i=1}^{N} x_i^{(k)} * A_{k,i} * G'(t) = f_j * G'(t)$$

$(1 \leq j \leq N, -\infty < t < \infty)$.
where \( A_k = [A_{ki}] \). Integrating by parts yields

\[
\begin{align*}
    v_{ij}^{(r-1)} & \ast G'(t) + \sum_{k=1}^{r-1} \sum_{s=1}^{N} x_i^{(s-1)} \ast A_{ki} \ast G'(t) + \sum_{s=1}^{N} x_i \ast A_{0i} \ast G(t) \\
    &= f_{ij} \ast G'(t) \quad (1 \leq j \leq N, -\infty < t < \infty).
\end{align*}
\]

Repeating this convolution and integration process \( v \) times, we obtain

\[
(3.2) \quad x \ast B(t) = g(t) \quad (-\infty < t < \infty)
\]

where \( B = [B_{ij}] \) with

\[
B_{ij} = \delta_{ij}(G')^{v*} + \sum_{k=1}^{r-1} A_{ki} \ast (G')^{v*} \ast G^{(r-k)*} + A_{0i} \ast G^{*},
\]

\( \delta_{ij} \) denoting the Kronecker delta, and \( g = f \ast C \) where \( C = [C_{ij}] \) with \( C_{ij} = \delta_{ij}(G')^{v*} \). Here we have used \( A^{k} \) and \( a^{k} \) to denote the \( k \)-fold convolutions \( A \ast A \ast \cdots \ast A, a \ast a \ast \cdots \ast a \). Using \( H(G, \eta), H(G', \eta), H(A_k, \eta) \) (\( 0 < k < r-1 \)) and \( H(f, \eta), H(g, \eta) \), one easily checks that \( H(B, \eta) \) and \( H(g, \eta) \) are satisfied with \( \lim_{t \to \infty} g(t) = f(\infty)[(G(\infty))^{v}E] \). The definition of \( B \) and (3.1) imply

\[
\det \tilde{B}(\lambda) = [(\tilde{G}(\lambda))^{(Nv)} \det[P(i\lambda)] \quad (-\infty < \lambda < \infty);
\]

hence (2) and (3.2) have the same spectral sets. Also, \( x \) satisfies (T) since \( x \in \text{LAC}(-\infty, \infty) \) and \( \|x'\|_\infty < \infty \). Thus, applying Theorem 1 to (3.2) and using the values of \( g(\infty) \) and \( B(\infty) \), we find (1.3) holds with \( \gamma_m \in C^N \) (\( 1 \leq m \leq n \)) and \( \eta(t) \to 0 \) as \( t \to \infty \). To show \( \eta \) satisfies (1.6), note that (the analogue for \( (-\infty, \infty) \) of) [2, Theorem 13a] implies

\[
x(t) = f(\infty)A_0(\infty)^{-1} + \sum_{m=1}^{n} c_m(t) \exp[i\lambda_m t] + \eta_1(t) \quad (-\infty < t < \infty)
\]

where \( \eta_1 \) satisfies (1.6), and the \( c_m \) satisfy

\[
(3.3) \quad c_m \in C^\infty(-\infty, \infty) \cap L^\infty(-\infty, \infty), \quad \lim_{t \to \infty} c_m^{(j)}(t) = 0 \quad (j = 1, 2, \cdots).
\]

Since \( \sum_{m=1}^{n} (c_m(t) - \gamma_m) \exp[i\lambda_m t] \to 0 \) as \( t \to \infty \), it follows from (3.3) that \( c_m(t) \to \gamma_m \) as \( t \to \infty \) (\( 1 \leq m \leq n \)). Thus \( \eta \) satisfies (1.6).

We remark that the technique used to prove Theorem 2 may also be employed to give a different proof of Theorem 13a in [2].
References


Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65201